Math 817–818 Qualifying Exam

May 2019

Instructions:

- Solve two problems from each of the three parts, for a total of six. Justify all of your answers.
- Each problem will be graded out of 20 points. Some problems have multiple parts, in which case the point values are given. You may assume the results of earlier parts, even if you do not solve them.
- If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

Section I: Group theory

Solve *two* of the following three problems.

(1) Let G be a group. A subgroup H of G is called *maximal* if $H \neq G$ (that is, H is a proper subgroup of G) and whenever K is another subgroup of G containing H, either $K = H$ or $K = G$.

Show that every nontrivial finitely generated group G possesses maximal subgroups.

(2) Let H be a subgroup of a group G. Recall that the centralizer of H is the subgroup of G defined by $C_G(H) = \{g \in G \mid gh = hg \ \forall h \in H\}$ and the normalizer of H is the subgroup of G defined by $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$.

(a) [**10 points**] Show that the centralizer $C_G(H)$ of H in G is a normal subgroup of the normalizer $N_G(H)$ of H in G.

(b) [**10 points**] Show that the quotient $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(H)$ of H.

(3) Let G be a finite group of order p^2q with $p < q$ prime numbers. Show that G is not a simple group.

Section II: Ring theory, module theory, and linear algebra

Solve *two* of the following three problems.

(4) Consider the subring $\mathbb{Z}[\sqrt{10}] = \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\}$ of $\mathbb{R}$. Show that 2 is irreducible but not prime in $\mathbb{Z}[\sqrt{10}]$.

Hint: Consider the function $N : \mathbb{Z}[\sqrt{10}] \to \mathbb{Z}, N(a + b\sqrt{10}) = a^2 - 10b^2$.

(5) Recall that a $\mathbb{Z}$-module M is called torsion-free if its torsion submodule is $\mathrm{Tor}(M) = \{0\}$, where $\mathrm{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some } r \in \mathbb{Z} \setminus \{0\}\}$.

Consider the $\mathbb{Z}$-module

$$M = \frac{\mathbb{Z} \oplus \mathbb{Z}}{\langle (7, 11) \rangle}, \text{ where } \langle (7, 11) \rangle = \{(7k, 11k) \mid k \in \mathbb{Z}\}.$$

Show that M is torsion-free.

(6) Let n be a positive integer. Consider the real vector space $V = \{p \in \mathbb{R}[x] \mid \deg(p) \leq n\}$ and the linear transformation $T : V \to V, T(p) = p'$, where $p'(x)$ is the derivative of $p(x)$.

- **(a)** [**7 points**] Find the characteristic polynomial and the minimal polynomial for T.
- **(b)** [**6 points**] Find the invariant factors and the elementary divisors for T.
- **(c)** [**7 points**] Find the rational canonical form and the Jordan canonical form for T.

Provide justification for each of your findings above.

Section III: Field theory and Galois theory

Solve *two* of the following three problems.

(7) Let E, K, and L be subfields of a field F with $E \subseteq K$ and $E \subseteq L$. Let $k = [K : E]$ and $\ell = [L : E]$. Recall that KL denotes the smallest (with respect to containment) subfield of F which satisfies $K \subseteq KL$ and $L \subseteq KL$.

- **(a)** [**10 points**] Show that $[KL : E] \leq k\ell$.
- **(b)** [**5 points**] Show that if $\gcd(k, \ell) = 1$ then $[KL : E] = k\ell$.
- **(c)** [**5 points**] Give an example satisfying $[KL : E] < k\ell$.

(8) Let $F \subseteq L$ be fields and let $x, y \in L$ be algebraic elements over F. Prove that $x + y$ and xy are also algebraic elements of L over F.

(9) Consider $f(x) = x^5 - 20x - 2 \in \mathbb{Q}[x]$. This polynomial has exactly three real roots, a fact that you may use without proof.

- **(a)** [**5 points**] Show that f is irreducible in $\mathbb{Q}[x]$.
- **(b)** [**15 points**] Let L be a splitting field of f over $\mathbb{Q}$. Show that L/Q is a Galois extension and find the isomorphism class of the Galois group $\mathrm{Gal}(L/\mathbb{Q})$.