Math 817–818 Qualifying Exam

May 2022

Instructions:

- Solve two problems from each of the three parts, for a total of six. Justify all of your answers.
- Each problem will be graded out of 20 points. For problems with multiple parts the point values for each part are given. You may assume the results of earlier parts, even if you do not solve them.
- If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.
- Please write on only one side of each page and number your pages across all problems.

Section I: Group theory

Solve *two* of the following three problems.

(1) Let G be a finite p-group for some prime p and $N \neq \{1\}$ a normal subgroup of G. Prove that $N \cap \mathrm{Z}(G) \neq \{1\}$, where $\mathrm{Z}(G)$ is the center of G.

(2) A non-abelian group of order 27.

(a) (12 points) Prove there exists a non-abelian group of 27. (Hint: Use a semi-direct product.)

(b) (8 points) Find, with justification, a presentation of the group you found in part (a).

(3) Let G be a group of order $2^5 \cdot 7^3$. Prove that G is not simple.

Section II: Rings, modules and linear algebra

Solve *two* of the following three problems.

(4) Let R be a commutative ring with identity, and assume $1 \neq 0$. Let I and J be ideals such that $I + J = R$.

(a) (8 points) Prove $IJ = I \cap J$.

(b) (12 points) Prove the following special case of the Chinese Remainder Theorem: There is an isomorphism of rings of the form $R/(I \cap J) \cong R/I \times R/J$.

(5) Prove the Rank-Nullity Theorem: If F is a field, V and W are a finite dimensional F-vector spaces, and $g : V \to W$ is an F-linear transformation, then $\mathrm{rank}(g) + \mathrm{nullity}(g) = \dim(V)$. (Recall the rank of g is the dimension of its image and the nullity of g is the dimension of its kernel.)

(6) Let F be a field and n a positive integer. We say an $n \times n$ matrix A with entries in F is *unipotent* if $A - I_n$ is nilpotent (i.e., $(A - I_n)^k = 0$ for some $k \geq 1$). For the field $F = \mathbb{Q}$, find (with complete justification) the number of similarity classes of 4×4 unipotent matrices and give an explicit representative for each class.

(See next page.)

Section III: Fields and Galois theory

Solve *two* of the following three problems.

(7) Let L be the splitting field of $x^5 - 11$ over $\mathbb{Q}$

(a) (10 points) Find, with justification, the degree of L over $\mathbb{Q}$.

(b) (10 points) Let $F = \mathbb{Q}(\zeta)$ where $\zeta = e^{2\pi i/5}$, a primitive 5-th root of unity. Prove $x^5 - 11$ is irreducible in $F[x]$.

(8) Let L be the splitting field of $x^4 - 2022$ over $\mathbb{Q}$. Prove there exists a unique intermediate field $\mathbb{Q} \subseteq K \subseteq L$ such that $[K : \mathbb{Q}] = 4$ and $\mathbb{Q} \subseteq K$ is a Galois extension.

(9) Assume $F \subseteq L$ is an algebraic field extension such that every non-constant polynomial in $F[x]$ splits completely into linear factors in $L[x]$. Prove L is an algebraic closure of F.