Math 817–818 Qualifying Exam

January 2017

Instructions:

- Solve *two* problems from each of the three parts, for a total of *six*.
- Each problem will be graded out of 20 points. Some problems have multiple parts, in which case the point values are given. You may assume the results of earlier parts, even if you do not solve them.
- If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.
- Justify all of your answers.

Section I: Group Theory

Do *two* of the following three problems.

1. Prove that no group of order $2^m \cdot 5$ with $m \geq 1$ is simple.
2. Let G be a group of order p^n where p is a prime and $n \geq 1$.
   (a) (10 points) Prove the center of G is not trivial.
   (b) (10 points) Prove there exists a subgroup of order p^j for each j satisfying $0 \leq j \leq n$.
3. Let p be any positive prime integer. Prove that the number of groups of order $3 \cdot p$, up to isomorphism, is exactly
$$\begin{cases} 2, & \text{when } p = 2,\ p = 3 \text{ or } p \equiv 1 \pmod 3\text{, and} \\ 1, & \text{otherwise.} \end{cases}$$

Section II: Field Theory and Galois theory

Do *two* of the following three problems.

4. Let L/F be a finite Galois field extension of degree 45. Prove there exists a unique intermediate field E (i.e., $F \subseteq E \subseteq L$) such that $[E : F] = 5$.
5. Let L be the splitting field of the polynomial $x^7 - 18$ over $\mathbb{Q}$. Give, with full justification, a presentation for the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ that has two generators.
6. Assume $F \subseteq L$ is a finite extension of fields and that the characteristic of F is p, where p is a prime. Prove that if there exists an element $\alpha \in L \setminus F$ such that $\alpha^p \in F$, then $\# \mathrm{Aut}(L/F) < [L : F]$. You may use, without proof, the fact that $\# \mathrm{Aut}(K/E) \leq [K : E]$ for any finite extension of fields $E \subseteq K$.

(See next page for part III.)

Section III: Ring theory, Module theory and Linear Algebra

Do *two* of the following three problems.

7. Let R be a PID and S a multiplicatively closed subset of R such that $0 \notin S$. Prove that $S^{-1}R$ is also a PID. *Tip*: Given an ideal I of $S^{-1}R$, consider $I \cap R$.

8. Find, with justification, all the ideals of the ring $\mathbb{Z}[x]$ that contain the ideal $I = (3, x^3 + x + 1)$.

9. Let F be a field, V a finite dimensional F-vector space, and $T : V \to V$ an F-linear operator. Prove T is diagonalizable if and only if its mimimum polynomial factors into distinct linear factors.