Math 817–818 Qualifying Exam

May 2017

Instructions:

- Solve *two* problems from each of the three parts, for a total of *six*.
- Each problem will be graded out of 20 points. Some problems have multiple parts, in which case the point values are given. You may assume the results of earlier parts, even if you do not solve them.
- If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.
- Justify all of your answers.

Section I: Group Theory

Do *two* of the following three problems.

1. Let G be a group of order $p^m q$ where p and q are distinct primes and $m \geq 1$. Assume that G has at least one element of order p^m. Let S be the set of all elements of G of order p^m, that is $S = \{x \in G : |x| = p^m\}$.
   (a) (5 points) Prove that G acts on S by conjugation.
   (b) (15 points) Prove that if $Z(G) = \{e\}$ then q divides $\#S$.
2. Prove there are exactly three groups, up to isomorphism, of order 75.
3. Prove that no group of order $2^m \cdot 7$ with $m \geq 1$ is simple

Section II: Ring theory, Module theory and Linear Algebra

Do *two* of the following three problems.

4. (a) (10 points) Prove that in a UFD an element p is irreducible if and only if the ideal (p) is prime.
   (b) (10 points) Prove that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.
5. Let F be a field and $f(x) \in F[x]$ a monic polynomial of degree n. Prove: all matrices in $\mathcal{M}_{n\times n}(F)$ having characteristic polynomial $f(x)$ are similar if and only if the irreducible factorization of $f(x)$ has no repeated factors.
6. Make $\mathbb{R}^3$ into a $\mathbb{R}[x]$-module by letting $f(x)\mathbf{v} = f(A)\mathbf{v}$ for any vector $\mathbf{v}$ in $\mathbb{R}^3$, where A is the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}.$$

   Recall that the $\mathbb{R}[x]$-module $\mathbb{R}^3$ described above is isomorphic to $\mathbb{R}[x]^3/\mathrm{Im}(\varphi)$, with $\varphi : \mathbb{R}[x]^3 \to \mathbb{R}[x]^3$ given by $\varphi(\mathbf{v}) = (Ix - A)\mathbf{v}$. Find a polynomial $p(x) \in \mathbb{R}[x]$ so that $\mathbb{R}^3 \cong \mathbb{R}[x]/(p(x))$ as $\mathbb{R}[x]$-modules.

Please turn the page for section III.

Section III: Field Theory and Galois theory

Do *two* of the following three problems.

7. Let $F \subseteq L$ be a finite extension of fields and let $f(x) \in F[x]$ be an irreducible polynomial of degree d. Prove that, if $\gcd(d, [L : F]) = 1$, then $f(x)$ is also irreducible in $L[x]$.

8. Let $F \subseteq L$ be a field extension with L algebraically closed. Consider the set

$$K = \{\alpha \in L : f(\alpha) = 0 \text{ for some monic polynomial } f(x) \in F[x]\}.$$

   (a) (10 points) Show that K is a field.

   (b) (10 points) Show that K is algebraically closed.

9. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible cubic (degree 3) polynomial having exactly one real root. Let L be the splitting field of $f(x)$ over $\mathbb{Q}$. Show that $\mathrm{Gal}(L/\mathbb{Q}) \cong S_3$.