Math 817-818 Qualifying Exam

Jan 12th, 2022

Instructions:

- Solve two problems from each of the three parts, for a total of six questions. Justify all of your answers.
- Each problem will be graded out of 20 points. For problems with multiple parts you may assume the results of earlier parts, even if you do not solve them.
- If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.
- Please write on only one side of each page and number your pages across all problems.

Section I: Group Theory

Question 1. Let G be a group, and $H, K \leq G$ be subgroups of G.

a) Give an example of G, H, and K such that HK is not a subgroup of G.

b) Suppose now that $H \trianglelefteq G$ and $[G : H] = p$, where p is prime. Prove that either $K \leq H$ or $G = KH$.

Question 2. Let G be a group of order $2835 = 3^4 \cdot 5 \cdot 7$.

a) Show that there are at most two options for n_3, the number of Sylow 3-subgroups of G, and list them.

b) Prove that G is not simple.

Question 3.

a) Let K be a subgroup of a group G. Prove that $K \trianglelefteq G$ if and only if there is a group H and a homomorphism $\phi : G \to H$ such that $K = \ker(\phi)$.

b) Let G be the group of all 2×2 matrices with entries from $\mathbb{Z}$ having determinant 1. Let p be a prime number, and take K to be the subset of G consisting of all $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a \equiv d \equiv 1 \pmod{p}$ and $b \equiv c \equiv 0 \pmod{p}$. Prove that K is a normal subgroup of G.

Section II: Rings, Modules, and Linear Algebra

Question 4.

a) Prove that a finite integral domain D must be a field.

b) Prove that if R is a commutative ring and $P \subseteq R$ is a prime ideal such that P has finite index as an additive subgroup of R, then P is a maximal ideal. Give an example to show that this implication may fail if the finite index assumption is dropped.

Question 5. Let F be a field, V and W finite dimensional F-vector spaces, and $g : V \to W$ an F-linear transformation.

a) Prove that there exists bases of V and W such that the matrix representing g with respect to these bases has the form
$$\begin{bmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{bmatrix},$$
where $I_{r\times r}$ is the $r \times r$ identity matrix and all the 0's denote zero matrices of the appropriate size.

b) Prove that the number r appearing in part a) is independent of choice of bases; that is, if, for another pair of bases of V and W, the matrix representing g has the form
$$\begin{bmatrix} I_{r'\times r'} & 0 \\ 0 & 0 \end{bmatrix}$$
then $r' = r$.

Question 6. Let F be any field.

a) Let A and B be two 3×3 matrices with entries in F. Prove A and B are similar if and only if they have the same characteristic polynomial and the same minimum polynomial.

b) Show, by way of an example with justification, that the previous part would become false if "3×3" were replaced by "4×4".

c) Give an example of a field F and two 3×3 matrices with entries in F having the same minimum polynomial that are *not* similar.

Section III: Fields and Galois Theory

Question 7. Let E be the splitting field of $x^4 + 5$ over $\mathbb{Q}$.

a) Prove, by adding two appropriate roots of $x^4 + 5$ or otherwise, that there exists $\mathbb{Q} \subseteq F \subseteq E$ such that $F \subseteq \mathbb{R}$ and $[F : \mathbb{Q}] = 4$.

b) Determine, with justification, $[E : \mathbb{Q}]$.

Question 8. Let F be the splitting field over $\mathbb{Q}$ of the polynomial $x^3 - 2$. Prove that the Galois group $\mathrm{Gal}(F : \mathbb{Q})$ is isomorphic to S_3.

Question 9.

a) Let F be a field and $f \in F[x]$ be irreducible. Prove the $F[x]/(f)$ is a field.

b) Give an explicit construction (with justification) of a field of size 16. [You may use without proof that the unique irreducible quadratic in $\mathbb{F}_2[x]$ is $x^2 + x + 1$.]