Math 817-818 Qualifying Exam

May 2021

Instructions:

- Solve two problems from each of the three parts, for a total of six questions. Justify all of your answers.
- Each problem will be graded out of 20 points. For problems with multiple parts the point values for each part are given. You may assume the results of earlier parts, even if you do not solve them.
- If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.
- Please write on only one side of each page and number your pages across all problems.

Section I: Group Theory

Question 1. Let G be a group (not necessarily finite) and H a nonempty subset of G that is closed under multiplication. Suppose that for all $g \in G$ we have $g^2 \in H$. Prove the following:

a) (7 points) H is a subgroup of G
b) (7 points) H is normal
c) (6 points) G/H is abelian.

Question 2. Construct a nonabelian group of order 21 and give a presentation for your group (with justification).

Question 3. Let G be a group and let n_p be the number of Sylow p-subgroups of G, where p is a prime dividing the order of G.

a) (10 points) Prove that if G is simple then $|G| \bigm| n_p!$.
b) (10 points) Deduce that there is no simple group of order 1,000,000.

Section II: Rings, Modules, and Linear Algebra

Question 4.

a) (14 points) Suppose that A is an integral domain that contains a field F as a subring, and that moreover A is finite dimensional as an F-vector space. Prove that A is a field.
b) (6 points) Give an example to show that the finite dimension condition is necessary.

Question 5. Suppose that $T : V \to V$ is a linear map, where V is a finite dimensional $\mathbb{C}$-vector space. Fix a polynomial $p \in \mathbb{C}[x]$.

a) (10 points) Prove that if λ is an eigenvalue of T then $p(\lambda)$ is an eigenvalue of $p(T)$.
b) (10 points) Prove conversely that if μ is an eigenvalue of $p(T)$ then there exists an eigenvalue λ of T such that $\mu = p(\lambda)$.

Question 6. Consider the $\mathbb{C}$-vector space $V = \{p \in \mathbb{C}[x] : \deg(p) \leq n-1\}$. [You may assume without proof that V is n-dimensional.] Consider the following linear maps $V \to V$:

$$D : V \to V \qquad\qquad T : V \to V$$
$$p \mapsto p' \qquad\qquad p \mapsto x\,p'$$

(where p' denotes the derivative of p). Determine the Jordan normal form of

a) (10 points) D
b) (10 points) T

Section III: Fields and Galois Theory

Question 7. Assume that $F \subseteq K$ is a finite extension of fields of degree $n = [K : F]$.

a) (15 points) Prove that if $f \in F[x]$ is irreducible of degree d and $\gcd(d, n) = 1$ then f remains irreducible when regarded as an element of the ring $K[x]$.

b) (5 points) Show, by means of an explicit example with justification, that the statement in part a) would become false if the assumption that $\gcd(n, d) = 1$ were omitted.

Question 8. Suppose that $F \subseteq L$ is a finite Galois extension with Galois group G, and that $\alpha \in L$. Prove that $L = F(\alpha)$ if and only if the images of α under elements of G are distinct.

Question 9. Let $p = x^6 + 3 \in \mathbb{Q}[x]$, and let L be its splitting field over $\mathbb{Q}$. Prove that if α is a root of p then $L = \mathbb{Q}(\alpha)$. Hence (or otherwise) determine $[L : \mathbb{Q}]$. [Hint: Consider the value of α^3.]