Math 817–818 Qualifying Exam

January 2023

Instructions:

- Solve two problems from each of the three parts, for a total of six. Justify all of your answers.
- Each problem will be graded out of 20 points. For problems with multiple parts the point values for each part are given. You may assume the results of earlier parts, even if you do not solve them.
- If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.
- Please write on only one side of each page and number your pages across all problems.

Section I: Group theory

Solve *two* of the following three problems.

(1) Let G be a finite group and let H be a subgroup of G of index p, where p is the smallest prime dividing the order of G. Prove that H is a normal subgroup of G.

Hint: Use an action of G on the set of cosets of H.

(2) (a) (10 points) Give two groups of order 21 which are not isomorphic, with justification.
(b) (10 points) Prove that there are at most three groups of order 21 up to isomorphism.

(3) Let G be a finite abelian group and call the exponent of G the smallest positive integer n which satisfies $g^n = e_G$ for all $g \in G$, where e_G is the identity element of G.

(a) (10 points) Give, with justification, a formula for the exponent of G in terms of the invariant factors of G. (As a reminder for those who confuse "invariant factors" and "elementary divisors": the invariant factors form a list in which each one divides the next.)
(b) (10 points) For p prime, determine the exponent of the abelian group $((\mathbb{Z}/p) \setminus \{0\}, \cdot)$ and use it to show that this group is cyclic.

Section II: Rings, modules and linear algebra

Solve *two* of the following three problems.

(4) Prove that in a principal ideal domain a nonzero ideal is prime if and only if it is maximal.

(5) Let I be a nonzero ideal of the ring of Gaussian integers $\mathbb{Z}[i]$. Prove that the quotient ring $\mathbb{Z}[i]/I$ is finite.

(6) Determine, with justification, if the following two matrices with complex entries

$$A = \begin{bmatrix} 0 & -4 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

are similar.

(See next page.)

Section III: Fields and Galois theory

Solve *two* of the following three problems.

(7) Let $K \subseteq L$ be a finite extension of fields and assume $f(x)$ is a polynomial with coefficients in K that is irreducible in the ring $K[x]$. Prove $f(x)$ remains irreducible when regarded as an element of the ring $L[x]$ provided $[L : K]$ is relatively prime to the degree of $f(x)$.

(8) Let L be the splitting field of $x^3 - 2$ over $\mathbb{Q}$.

   (a) (10 points) Prove there is a unique intermediate field $\mathbb{Q} \subseteq K \subseteq L$ such that $[K : \mathbb{Q}] = 2$.

   (b) (10 points) Find, with justification, a primitive generator for the field K you found in part (a); that is, find an *explicit* element $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$.

(9) Let F be a field of characteristic $p > 0$ and let $F \subseteq L$ be an extension of fields, and assume there is an element $a \in L$ such that $a^p \in F$ but $a \notin F$. Prove $\# \operatorname{Aut}(F(a)/F) < [F(a) : F]$