Math 817–818 Qualifying Exam

January 2021

Instructions:

- Solve two problems from each of the three parts, for a total of six. Justify all of your answers.
- Each problem will be graded out of 20 points. For problems with multiple parts the point values for each part are given. You may assume the results of earlier parts, even if you do not solve them.
- If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.
- Please write on only one side of each page and number your pages across all problems.
- Contact Mark Walker at (402) 430-6463 if you have any questions.

Section I: Group theory

Solve *two* of the following three problems.

(1) Let $G = A_7$ and S be the set of elements of G of order 7. Prove that S is *not* a conjugacy class of G.

(2) Let G be a group acting transitively on a set S. For $a \in S$, let $G_a = \{g \in G \,|\, ga = a\}$ be the stabilizer of a under the action from G.

- **(a)** **[10 pts]** Let $a, b \in S$. Prove there exists $g \in G$ such that $G_b = gG_ag^{-1}$.
- **(b)** **[10 pts]** Suppose S has more than one element and that G is finite. Prove that there exists $g \in G$ which has no fixed point, that is, for all $a \in S$, $ga \neq a$.

(3) Let G be a group of order $175 = 3^2 \cdot 5^2$ and suppose G contains an element of order 25. Prove that G is abelian.

Section II: Ring theory and module theory

Solve *two* of the following three problems.

(4) Let R be a commutative ring with identity, and assume $1 \neq 0$.

- **(a)** **[10 pts]** Prove that every maximal ideal of R is a prime ideal.
- **(b)** **[10 pts]** Assume R is a finite ring. Prove that every prime ideal is a maximal ideal.

(More on next page.)

(5) Let R be an integral domain and let M be an R-module. Recall that a subset S of M is *linearly independent* if whenever $\sum_i r_i s_i = 0$ for ring elements r_i and elements $s_i \in S$, we must have $r_i = 0$ for all i. We say S is *maximally linearly independent* if it is linearly independent and it is not properly contained in a linear independent subset of M. Finally, recall that we say an R-module P is *torsion* if for all $p \in P$, $rp = 0$ for some non-zero $r \in R$.

- **(a)** **[10 pts]** Suppose S is a linearly independent subset of M and let N be the submodule of M generated by S. Prove it is maximally linearly independent if and only if M/N is torsion.
- **(a)** **[10 pts]** Prove that if for every module M every maximally independent subset of M generates M as an R-module, then R must be a field.

(6) Let R be a commutative ring with $1 \neq 0$ and let $f : R^m \twoheadrightarrow R^n$ be a surjective R-module homomorphism, with m and n integers. Prove $m \geq n$.

Section III: Linear algebra and field theory

Solve *two* of the following three problems.

(7) On canonical forms

- **(a)** **[10 pts]** Consider the $\mathbb{Q}[x]$-module

$$M = \frac{\mathbb{Q}[x]}{(x^4 - 1)} \oplus \frac{\mathbb{Q}[x]}{(x^2(x-1))}$$

and let V the $\mathbb{Q}$-vector space obtained from M by restriction of scalars along the evident inclusion $\mathbb{Q} \subseteq \mathbb{Q}[x]$ and let $t : V \to V$ be the $\mathbb{Q}$-linear transformation given as multiplication by x. Find, with justification, the rational canonical form of t.

- **(a)** **[10 pts]** Consider the $\mathbb{C}[x]$-module

$$N = \frac{\mathbb{C}[x]}{(x^4 - 1)} \oplus \frac{\mathbb{Q}[x]}{(x^2(x-1))}$$

and let W the $\mathbb{C}$-vector space obtained from N by restriction of scalars along $\mathbb{C} \subseteq \mathbb{C}[x]$ and let $t : W \to W$ be the $\mathbb{C}$-linear transformation given as multiplication by x. Find, with justification, the Jordan canonical form of t.

(8) Let $f(x) = x^p - 5 \in \mathbb{Q}[x]$ where p is an odd prime, and let L be the splitting field of $f(x)$ over $\mathbb{Q}$. Find, with justification, $[L : \mathbb{Q}]$.

(9) Let F be a field, V an F-vector space, and W a subspace of V. A subspace U of V is called a *complement* of W in V if V is the internal direct sum of W and U; that is, $V = W \oplus U$.

- **(a)** **[10 pts]** Prove that for every V and W as above, W has at least one complement in V.
- **(a)** **[10 pts]** Prove that if U is a complement of W in V and V is finite dimensional, then $\dim_F(V) = \dim_F(W) + \dim_F(U)$ (where $\dim_F$ denotes the dimension of an F-vector space).