Math 817-818 Qualifying Exam
January 2019

Instructions:

- Solve two problems from each of the three parts, for a total of six. Justify all of your answers.
- Each problem will be graded out of 20 points. Some problems have multiple parts, in which case the point values are given. You may assume the results of earlier parts, even if you do not solve them.
- If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

Section I: Group theory

Solve *two* of the following three problems.

(1) Let G be a group. A subgroup H of G is called a *characteristic* subgroup of G if $\alpha(H) = H$ for every automorphism α of G. Show that if H is a characteristic subgroup of N and N is a normal subgroup of G, then H is a normal subgroup of G.

(2) Let G be a group acting on a set A, and let H be a subgroup of G satisfying that the induced action of H on A is transitive (that is, for all $a, b \in A$ there is an $h \in H$ with $h \cdot a = b$). Let $t \in A$, and let $Stab_G(t)$ be the stabilizer of t in G. Show that $G = HStab_G(t)$.

(3)

(a) [**10 points**] Let G be a simple group of order 60. Determine the number of elements of G of order 5.

(b) [**10 points**] Show that there is no simple group of order 30.

Section II: Ring theory, module theory, and linear algebra

Solve *two* of the following three problems.

(4) Let d be a square-free integer. The ring $\mathbb{Q}(\sqrt{d})$ is the subring of $\mathbb{C}$ defined by

$$\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}.$$

Show that there is a ring isomorphism $\mathbb{Q}[x]/(x^2 - d) \cong \mathbb{Q}(\sqrt{d})$.

(5) Let A be a $\mathbb{Z}$–module and let n be any integer. Show that there is a $\mathbb{Z}$-module isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong A_n, \text{ where } A_n = \{a \in A \mid na = 0\}$$

is a submodule of A. (Note: You may use the fact that A_n is a $\mathbb{Z}$–module without proof.)

(6) Determine all similarity classes of matrices with entries in $\mathbb{Q}$ with characteristic polynomial $(x^4 - 1)(x^2 - 1)$. Provide an explicit representative for each of these similarity classes.

Section III: Field theory and Galois theory

Solve *two* of the following three problems.

(7) Let p be a prime integer and let α be a root of the polynomial $x^3 - p$.

- **(a)** [**14 points**] Find, with justification, the degree of the field extension $\mathbb{Q}(\alpha, i)$ over $\mathbb{Q}$.
- **(b)** [**6 points**] Deduce that the polynomial $x^3 - p$ is irreducible in $\mathbb{Q}(i)[x]$.

(8)

- **(a)** [**10 points**] Show that any finite extension of fields K/F is algebraic.
- **(b)** [**10 points**] Let $\overline{\mathbb{Q}}$ denote the subfield of $\mathbb{C}$ consisting of all the complex numbers which are algebraic over $\mathbb{Q}$. (You may use that $\overline{\mathbb{Q}}$ is a field without proof.) Show that $\overline{\mathbb{Q}}/\mathbb{Q}$ is an algebraic extension, but not a finite extension.

(9) Let K be a Galois extension of F with $|Gal(K/F)| = 20$.

- **(a)** [**10 points**] Prove that there exists a subfield E of K containing F with $[E : F] = 5$.
- **(b)** [**10 points**] Determine whether there must also exist a subfield L of K containing F with $[L : F] = 10$.