Math 817–818 Qualifying Exam

January 2020

Instructions:

- Solve two problems from each of the three parts, for a total of six. Justify all of your answers.
- Each problem will be graded out of 20 points. For problems with multiple parts the point values for each part are given. You may assume the results of earlier parts, even if you do not solve them.
- If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.
- Please write on only one side of each page and number your pages across all problems.

Section I: Group theory

Solve *two* of the following three problems.

(1) Let G be a finite group.

- **(a)** [**12 pts**] Suppose every subgroup of G is normal. Prove that given any positive divisor d of $|G|$ there exists a subgroup of G of order d.
- **(b)** [**8 pts**] Give an example, with justification, of a finite group G and a positive divisor d of $|G|$ such that G has no subgroup of order d.

(2) Let G be a group of order p^n, for some prime p, acting on a finite set X.

- **(a)** [**10 pts**] Suppose p does not divide $|X|$. Prove that there exists some element of X fixed by all elements of G.
- **(b)** [**10 pts**] Suppose G acts faithfully on X. Prove that $|X| \geq np$.

(3) Prove that any group of order $3^2 \cdot 11 \cdot 17$ is abelian.

Section II: Ring theory, module theory, and linear algebra

Solve *two* of the following three problems.

(4) Let R be a commutative domain.

- **(a)** [**5 pts**] State the definition for R to be a Euclidean domain.
- **(b)** [**15 pts**] Prove that in a Euclidean domain every ideal is principal.

(5) Let R be a commutative ring in which every element x satisfies $x^n = x$ for some $n > 1$. Show that every prime ideal in R is maximal.

(6) Let n be a positive integer and let $J = J_{2n}(0)$ be the Jordan block matrix of size $2n \times 2n$ with eigenvalue 0 in $M_{2n\times 2n}(\mathbb{C})$.

- **(a)** [**10 pts**] Find the minimal polynomials for J and for J^2, with justification.

(b) **[10 pts]** Find the Jordan canonical form of J^2, with justification.
Hint: consider the kernel of J^2.

Section III: Field theory and Galois theory

Solve *two* of the following three problems.

(7) Let L/F be a field extension and let $K := \{a \in L \mid a \text{ is algebraic over } F\}$. Show that if L is algebraically closed, then K is algebraically closed.

(8) Let n be a positive integer and let p be a prime integer. Consider the polynomial

$$q(x) = x^{p^n} - x \in (\mathbb{Z}/p)[x]$$

and define K to be the splitting field of q over $\mathbb{Z}/p$.

- **(a)** **[10 points]** Show that the set E of roots of $q(x)$ in K is a subfield of K.
- **(b)** **[10 points]** Show that K has exactly p^n elements.

(9) Consider $f(x) = x^5 - 4x - 2 \in \mathbb{Q}[x]$. This polynomial has exactly three real roots, a fact that you may use without proof.

- **(a)** **[5 points]** Show that f is irreducible in $\mathbb{Q}[x]$.
- **(b)** **[15 points]** Let L be a splitting field of f over $\mathbb{Q}$. Show that L/Q is a Galois extension with Galois group $\mathrm{Gal}(L/\mathbb{Q})$ isomorphic to the symmetric group S_5.