Math 817–818 Qualifying Exam

June 2020

Instructions:

- Solve two problems from each of the three parts, for a total of six. Justify all of your answers.
- Each problem will be graded out of 20 points. For problems with multiple parts the point values for each part are given. You may assume the results of earlier parts, even if you do not solve them.
- If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.
- Please write on only one side of each page and number your pages across all problems.
- Contact Mark Walker at (402) 430-6463 if you have any questions.

Section I: Group theory

Solve *two* of the following three problems.

(1) Let p be a prime.

(a) [**10 pts**] Prove that every group of order p^2 is abelian. (Provide as many details as you can, but you may use main theorems – such as Lagrange's Theorem or the Class Equation – without proof.)

(b) [**10 pts**] Prove that there exists a non-abelian group of order p^3 for any prime p. (You may use facts about automorphism groups here without proof, although they should be clearly stated.)

(2) Let G be a group of order $2m$ for some odd integer $m > 1$. Let $\text{Perm}(G)$ be the group of permutations on the elements of G (i.e., the group of bijective functions from G to G). You may use without proof that $\text{Perm}(G) \cong S_{2m}$.

(a) [**5 pts**] Let $f : G \to \text{Perm}(G)$ be given by $f(g) = \lambda_g$ for all $g \in G$, where $\lambda_g : G \to G$ is defined by $\lambda_g(a) = ga$ for all $a \in G$. Prove that f is an injective group homomorphism.

(b) [**7 pts**] Prove that if $g \in G$ is an element of order 2 then λ_g is a product of m transpositions.

(c) [**8 pts**] Prove G contains a subgroup of index 2 and is consequently not simple. *Tip*: Consider the inverse image under f of the subgroup of $\text{Perm}(G)$ consisting of the even permutations.

(3) Let G be a finite group and p a prime dividing the order of G. Let P be a Sylow p-subgroup.

(a) [**10 pts**] Suppose Q is a Sylow p-subgroups distinct from P. Prove that PQ is not a subgroup of G. (Recall that $PQ = \{pq \mid p \in P, q \in Q\}$.)

(b) [**10 pts**] Prove that $N_G(P) = N_G(N_G(P))$ where $N_G(H)$ denotes the normalizer in G of a subgroup H.

Section II: Ring theory and module theory

Solve *two* of the following three problems.

(4) Let R be a commutative ring with identity and I and J two ideals of R such that $I+J=R$. For this problem, do not assume the the statement (or any parts) of the Chinese Remainder Theorem.

- **(a)** [**10 pts**] Prove that $IJ = I \cap J$, where IJ is the ideal of R generated by the set $\{ab \mid a \in I, b \in J\}$.
- **(b)** [**10 pts**] Prove that the function $\phi : R \to R/I \times R/J$ given by $\phi(r) = (r+I, r+J)$ is a surjective ring homomorphism.

(5) Let R be a non-zero, unital ring, and let R^m and R^n be the standard free left R-modules of finite rank m and n. Assume there is an isomorphism of R-modules $R^m \cong R^n$.

- **(a)** [**10 pts**] Prove that if R is commutative then $m = n$. You may assume without justification that this holds in the special case when R is a field.
- **(b)** [**10 pts**] Show, by example, than m need not equal n if R is not assumed to be commutative.

(6) Find all the ideals of $\mathbb{Z}[x]$ that contain $(6, x^2+x+1)$, the ideal of $\mathbb{Z}[x]$ generated by 6 and x^2+x+1. For each such ideal, give an explicit list of generators, and determine whether the ideal is prime, maximal, or neither.

Section III: Linear algebra and field theory

Solve *two* of the following three problems.

(7) Suppose F is any field. Recall that a square matrix A with entries in F is *nilpotent* if $A^j = 0$ for some positive integer j.

- **(a)** [**10 pts**] Prove that if $A \in \mathrm{Mat}_{n\times n}(F)$ and A is nilpotent, then $A^n = 0$.
- **(b)** [**10 pts**] Find, with justification, the number of similarity classes of 5×5 nilpotent matrices with entries in F.

(8) Let L be the splitting field over $\mathbb{Q}$ of the polynomial $f(x) = x^5 + 2 \in \mathbb{Q}[x]$.

- **(a)** [**10 pts**] Find, with justification, $[L : \mathbb{Q}]$.
- **(b)** [**10 pts**] Note that $w = -\sqrt[5]{2}$ (where $\sqrt[5]{2}$ is the unique positive, real fifth root of 2) is one of the roots of $f(x)$. Prove $\mathrm{Aut}(\mathbb{Q}(w)/\mathbb{Q})$ is the trivial group.

(9) Assume F is field and let $f(x) \in F[x]$. Recall that $f(x)$ is *separable* if $f(x)$ has no repeated roots in an algebraic closure of F.

- **(a)** [**10 pts**] Assume $\mathrm{char}(F) = 0$. Prove that $f(x)$ is separable if and only if the irreducible factorization of $f(x)$ in $F[x]$ has no repeated factors.
- **(b)** [**10 pts**] Fix a prime integer p, let $\mathbb{F}_p$ be the field with p elements, and let F be the field of fractions of the polynomial ring $\mathbb{F}_p[y]$. Prove $x^p - y$ is irreducible in $F[x]$ but not separable.