# Math 817–818 Qualifying Exam

## May 31, 2023

**Instructions:**

- Solve two problems from each of the three parts, for a total of six. Justify all of your answers.
- Each problem will be graded out of 20 points. For problems with multiple parts the point values for each part are given. You may assume the results of earlier parts, even if you do not solve them.
- If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.
- Please write on only one side of each page and number your pages across all problems.

## Section I: Group theory

Solve *two* of the following three problems.

(1) Let $G$ be a group and let $H$ be a subgroup of $G$. The following sets are subgroups of $G$

$$N_G(H) = \{g \in G : gHg^{-1} \subseteq H\} \quad \text{and} \quad C_G(H) = \{g \in G : gh = hg, \forall h \in H\},$$

a fact which you may use without proof.

(a) (5 points) Prove that $C_G(H)$ is a normal subgroup of $N_G(H)$.
(b) (15 points) Prove that $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$, the group of automorphisms of $H$.

(2) (a) (10 points) Let $G$ be a group with center $Z(G)$. Prove that if the quotient group $G/Z(G)$ is cyclic, then $G$ is abelian.
(b) (10 points) Let $G$ be a non-abelian group of order 21. Find the number and the sizes of the conjugacy classes of $G$, with justification.

(3) Let $p \neq q$ be two prime integers. Prove that a group of order $p^2q$ is not simple.

## Section II: Rings, modules and linear algebra

Solve *two* of the following three problems.

(4) Give an example of a maximal ideal $\mathfrak{m}$ in $\mathbb{Z}[x]$, the ring of polynomials with integer coefficients in one variable $x$. Prove that your example, $\mathfrak{m}$, is a maximal ideal, including a justification that $\mathfrak{m} \neq \mathbb{Z}[x]$.

(5) Prove that a principal ideal domain $R$ satisfies the ascending chain condition on ideals: every chain of ideals of $R$, $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq I_{n+1} \subseteq \cdots$, has $I_n = I_{n+1}$ for sufficiently large integers $n$. Do not invoke any theorems that trivialize the proof.

*(See the next page for more problems.)*

(6) Let $A$ be an $n \times n$ matrix with entries in $\mathbb{Z}$, for some $n \geqslant 1$, and let $M$ be the abelian group presented by $A$; that is, let $M = \mathbb{Z}^n/N$ where $N$ is the image of the map $\mathbb{Z}^n \to \mathbb{Z}^n$ given by $v \mapsto Av$. Prove $M$ is an infinite abelian group if and only if $\det(A) = 0$.

## Section III: Fields and Galois theory

Solve *two* of the following three problems.

(7) (a) (10 points) Prove that every finite field extension is algebraic.

(b) (10 points) Prove that the converse of (a) is false.

(8) Let $f \in \mathbb{Q}[x]$ be an irreducible cubic (meaning $f$ is a degree 3 polynomial) with exactly one real root. Let $L$ be the splitting field of $f$ over $\mathbb{Q}$. Show that $\mathrm{Aut}(L/\mathbb{Q}) \cong S_3$.

(9) Let $F \subseteq L$ be a Galois extension of order $p^2$, where $p$ is a prime integer.

(a) (5 points) Show that for every intermediate field $F \subseteq E \subseteq L$, the extension $F \subseteq E$ is Galois.

(b) (15 points) Show that there must be either 1 or $p + 1$ intermediate fields $F \subsetneq E \subsetneq L$.