Math 817-818 Qualifying Exam
May 2018

Instructions:

- Solve two problems from each of the three parts, for a total of six. Justify all of your answers.
- Each problem will be graded out of 20 points. Some problems have multiple parts, in which case the point values are given. You may assume the results of earlier parts, even if you do not solve them.
- If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

Section I: Group theory

Do *two* of the following three problems.

(1) Let H and K be groups. Recall that for any group G, an automorphism of G is an isomorphism from G to G, and $Aut(G)$ denotes the group of automorphisms of G.

(a) [14 points] Show that the direct product group $Aut(H) \times Aut(K)$ is isomorphic to a subgroup of $Aut(H \times K)$.

(b) [6 points] Give an example, with justification, of groups H and K for which $Aut(H) \times Aut(K)$ is not isomorphic to $Aut(H \times K)$.

(2) Let K be a normal subgroup of a group H. Recall that a group G is *solvable* if there is a sequence of subgroups

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_j = G$$

for some $j \geq 0$ such that for all $i \in \{0, ..., j-1\}$ the quotient group N_{i+1}/N_i is abelian. Show that if both K and H/K are solvable groups, then H is solvable.

(3) Determine all of the groups of order 45, up to isomorphism.

Section II: Ring theory, module theory, and linear algebra

Do *two* of the following three problems.

(4) In the commutative ring $R = \mathbb{Z}[\sqrt{-3}]$, show that the element 2 is irreducible but not prime.

(5) Let R be a commutative ring (with 1) and let M be an R-module. Show that if $M \neq 0$ and the only submodules of M are 0 and M, then there is a maximal ideal I of R such that M is isomorphic to R/I.

(6) For the matrix $A = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}$ in $Mat_3(\mathbb{R})$:

(a) [12 points] Find the rational canonical form of A.

(b) [8 points] Determine whether or not A has a Jordan canonical form, and if so, find this form.

Section III: Field theory and Galois theory

Do *two* of the following three problems.

(7) Suppose that K/F is a finite extension of fields such that the degree $[K : F]$ is odd. Show that if $b \in K$, then $F(b) = F(b^2)$.

(8) Let L/F be an extension of fields and let $a, b \in L$. Show that

$$Aut(L/F(a,b)) = Aut(L/F(a)) \cap Aut(L/F(b)).$$

(9) Let K be the splitting field of $x^4 - 3$ over $\mathbb{Q}$.

(a) [10 points] Find a basis for K as a vector space over $\mathbb{Q}$.

(b) [10 points] Show that $Aut(K/\mathbb{Q})$ is not abelian.