Analysis Qualifying Exam **29 Jan 2000**

Do 6 of the following 8 questions.

1. Consider a sequence of functions h_n given by $h_n(x) = \begin{cases} 2nx & 0 \le x \le 1/(2n), \\ 2 - 2nx & 1/(2n) \le x \le 1/n, \\ 0 & x > 1/n. \end{cases}$

   (a) Prove that $h_n \to 0$ uniformly on $[\delta, 1]$ for all δ in $(0, 1]$, but that convergence is not uniform on $[0, 1]$.

   (b) Prove, if $f : [0, 1] \to \mathbb{R}$ is continuous, then $\lim_{n\to\infty} \int_0^1 f(h_n(t))\, dt = f(0)$.

2. Suppose x_n is a sequence of real numbers with $x_n \to x$ and $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of continuous functions with $f_n \to f$ uniformly. Prove that $f_n(x_n)$ converges to $f(x)$.

3. Define a function β on $[-1, 1]$ by $\beta(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x > 0. \end{cases}$

   Prove that a function $f : [-1, 1] \to \mathbb{R}$ is Riemann-Stieltjes integrable with respect to β if and only if $f(0) = \lim_{x\to 0^+} f(x)$. Also prove that, in this case, $\int_{-1}^{1} f\, d\beta = f(0)$.

4. Consider the series $f(x) = \sum_{n=0}^{\infty} \frac{1}{1 + n^2 x}$. Prove your answers to the following questions.

   (a) For which values of x does the series converge pointwise?

   (b) On which intervals is the convergence uniform?

   (c) Is f continuous when it converges?

5. Let (X, d) be a metric space and let E be a compact nonempty subset of X. If $\delta = \sup\{d(x, y) : x, y \in E\}$, show that there are $a, b \in E$ with $d(a, b) = \delta$.

6. (a) State the Mean Value Theorem.

   (b) Suppose f is continuous on $I = [a, b]$ and that its derivative exists in (a, b). Prove that if $f'(x) > 0$ for all x in (a, b), then f is increasing, i.e., if $x_1, x_2 \in I$ and $x_1 < x_2$, then $f(x_1) < f(x_2)$.

   (c) Show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$ and has a local minimum at c, then $f'(c) = 0$.

7. Consider the function $f(x) = \begin{cases} x & x \text{ irrational}, \\ 0 & x \text{ rational}. \end{cases}$

   (a) What is $\sup_{0 \leq x \leq 1} f(x)$? Prove your answer.

   (b) Let (x_n) be a sequence of real numbers. Define $\limsup_{n \to \infty} x_n$.

   (c) Let $x_n = 1 - \left(\frac{1}{\sqrt{2}} \right)^n$. What is $\limsup_{n \to \infty} f(x_n)$? Prove your answer.

8. (a) Let (x_n) be a sequence of real numbers. Suppose there is $r \in (0, 1)$ so that

$$|x_{n+2} - x_{n+1}| \leq r|x_{n+1} - x_n|.$$

   Prove that (x_n) converges.

   (b) Show that the following sequence converges and compute its limit.

$$a_1 = 1, \qquad a_n = 1 + \frac{1}{a_{n-1}} \quad \text{for } n > 1.$$