Answer 5 of the following 8 questions. All have equal weight.

1. (a) Using the definition of a limit, show from first principles that $\lim_{n\to\infty} \frac{3n^2+2}{n^2+5} = 3$.

   (b) Consider $f : \mathbb{R} \to \mathbb{R}$. Recall that $\limsup_{y\to x} f(y) = \inf_{\delta>0} \left(\sup_{0<|x-y|<\delta} f(y) \right)$.

   Prove that $\lim_{y\to x} f(y) = c$ if and only if $\limsup_{y\to x} |f(y) - c| = 0$.

2. Suppose that (a_n) and (b_n) are sequences of positive numbers and that $\lim_{n\to+\infty} a_n/b_n = L$ where $L \in (0, +\infty)$. Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

3. If A_1 and A_2 are disjoint closed subsets of a metric space, construct disjoint open subsets B_1 and B_2 with $B_1 \supseteq A_1$ and $B_2 \supseteq A_2$.

4. If $f : (0, 1] \to \mathbb{R}$ is uniformly continuous on $(0, 1]$, prove that f is bounded on $(0, 1]$.

5. (a) State the Mean Value Theorem.

   (b) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and f is differentiable at all points of (a, b) except possibly at $x_0 \in (a, b)$. If $\lim_{x\to x_0} f'(x)$ exists, show that $f'(x_0)$ exists and $f'(x_0) = \lim_{x\to x_0} f'(x)$. Hint: consider the intervals $[x_0, x_0 + h]$ and $[x_0 - h, x_0]$ and use part (a).

6. Let $P_n(x)$ be the Taylor polynomial of degree n at the origin for e^x.

   (a) Find an upper bound for $\{|e^x - P_9(x)| : x \in [0, 1]\}$.

   (b) Find h so that $\sup\{|e^x - P_9(x)| : x \in [0, h]\} < 10^{-6}$.

   (c) Find n so that $\sup\{|e^x - P_n(x)| : x \in [0, 1]\} < 10^{-6}$.

   Possibly useful fact: $9! = 362,880 < 4 \cdot 10^5$.

7. Define g by $g(x) = \begin{cases} -1 & x = 0 \\ x & 0 < x < 1 \\ 2 & x = 1 \end{cases}$ Compute the Riemann-Stieltjes integral $\int_0^1 \frac{1}{x+3}\, dg(x)$.

8. Consider a sequence of functions f_n on $[0, +\infty)$ given by $f_n(x) = n^2 x e^{-nx}$.

   (a) Prove that f_n does not converges uniformly on $[0, +\infty)$.

   (b) For all $\delta > 0$, prove that f_n converges uniformly on $[\delta, +\infty)$.