Answer 5 of the following 7 questions. All have equal weight.

1. (a) Define $\lim_{n\to\infty} x_n = L$.

   (b) From the definition, show that $\lim_{n\to\infty} \frac{n^2-n}{2n^2+3} = \frac{1}{2}$.

   (c) Using the definition, prove that if (a_n) and (b_n) are sequences with $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then $\lim_{n\to\infty} a_n b_n = LM$. State clearly any results that you use.

2. Let $\epsilon > 0$ and let f be a continuous function on $\mathbb{R}$. Define a new function G_ϵ by

$$G_\epsilon(x) = \int_x^{x+\epsilon} f(t)\,dt.$$

   (a) Show that G_ϵ has a continuous derivative and compute $G'_\epsilon(x)$.

   (b) Show that $\lim_{\epsilon\to 0} \frac{G_\epsilon(x)}{\epsilon}$ exists and find its value.

3. (a) Let Θ be a collection of pairwise disjoint open intervals of $\mathbb{R}$. Show that Θ is at most countable.

   (b) Show that the set of all increasing sequences of natural numbers (i.e., sequences $(n_1, n_2, \ldots)$ with $n_k \in \mathbb{N}$ and $n_{k+1} \geq n_k$ for all $k \in \mathbb{N}$) is uncountable.

4. (a) State carefully the Mean Value Theorem.

   (b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a continuous derivative on $[a, b]$ and is twice differentiable on (a, b). Prove that if $f(a) = f(b) = 0$ and $f'(a) = f'(b) = 0$, then there are x_1 and x_2 in (a, b) with $x_1 \neq x_2$ so that $f''(x_1) = f''(x_2)$.

5. (a) Does the series $\sum_{n=1}^{\infty} (-1)^n \sin \frac{3}{n}$ converge absolutely, converge conditionally, or diverge? Prove your answer.

   (b) For which real numbers x does the power series $\sum_{n=1}^{\infty} \frac{3^n}{n^3} x^n$ converge? Again, prove your answer.

6. (a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous. Let $f_n(x) = f\left(x + \frac{1}{n}\right)$. Prove that f_n converges uniformly to f on $\mathbb{R}$.

   (b) Does this remain true if f is just continuous? Prove it or provide a counterexample. (If you provide a counterexample, prove that your counterexample *is* a counterexample.)

7. (a) Suppose that X is a compact metric space, Y is a metric space and $f : X \to Y$ is one-to-one, onto, and continuous. Show that $f^{-1} : Y \to X$ is continuous.

   (b) Show, by means of an example, that the previous part is false if X is not compact.