MASTERS COMPREHENSIVE & PH.D. QUALIFYING EXAM
ANALYSIS: MATH 825/826 JANUARY 23, 2003

Instructions: *Answer 5 of the following 7 questions. Each question carries equal weight. If you work on more than five questions, clearly indicate which ones you want graded. Different parts of a question do not necessarily have the same weight. Use white paper and write on one side of the paper only.*

(1) (a) Let $\{a_k\}$ be a sequence of real numbers such that the series $\sum_{k=1}^{\infty} a_k$ is convergent and that the series $\sum_{k=1}^{\infty} a_k^2$ is divergent. Prove that the series $\sum_{k=1}^{\infty} a_k$ does not converge absolutely.

(b) Consider the series $\sum_{k=1}^{\infty} \frac{1}{1+z^k}$, where $z \in \mathbb{C}$. Prove that the series diverges for all $|z| \leq 1$; and the series converges absolutely for all $|z| > 1$.

(2) Let (X, d) be a compact metric space and $f : X \to \mathbb{R}$ is a continuous function on X.
(a) Prove that $f(X)$ is compact in $\mathbb{R}$.
(b) Prove that there exists a point $a \in X$ such that $f(a) = \sup f(X)$.

(3) (a) Assume that $\sum_{k=1}^{\infty} a_k$ is a convergent series of nonnegative real numbers. Prove that the series $\sum_{k=1}^{\infty} a_k^x$ converges uniformly on $[1, \infty)$.

(b) Prove: the series $\sum_{k=0}^{\infty} \frac{x^3}{(1+x^3)^k}$ converges uniformly on $[a, b]$ for every $0 < a < b$; but the convergence is not uniform on $[0, b]$ for any $b > 0$.

(4) Let f be given by: $f(x) = \sum_{n=1}^{\infty} \frac{|x|}{x^2+n^2}$.
(a) Show that f is well defined on $\mathbb{R}$.
(b) Prove that f is continuous on $\mathbb{R}$, but f is not differentiable at 0.

(5) (a) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Prove that there exists $c \in (a, b)$ such that

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c).$$

(b) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function on $[0, 1]$ with $f(0) = 0$. Prove that there exists a sequence of polynomials $\{Q_n\}$ such that $xQ_n(x) \to f(x)$ uniformly on $[0, 1]$.

(6) (a) Let α be monotone increasing function on $[a, b]$ and assume that α is continuous at some point $s \in [a, b]$. Prove that if $f(s) = 1$ and $f(x) = 0$ for $x \neq s$, then $f \in \mathcal{R}(\alpha)[a, b]$ (i.e., f is Riemann integrable with respect to α on $[a, b]$) and that $\displaystyle\int_a^b f d\alpha = 0$.

(b) Let α_n be a sequence of monotone increasing functions on $[a, b]$ and assume that f is a bounded function on $[a, b]$ such that $f \in \mathcal{R}(\alpha_n)[a, b]$ for every $n \in \mathbb{N}$. Prove that: if $\lim_{n\to\infty} \alpha_n(x) = 0$ for each $x \in [a, b]$, then $\displaystyle\int_a^b f d\alpha_n \to 0$, as $n \to \infty$.

(7) (a) Let $f_n : [a, b] \to \mathbb{R}$, $n = 1, 2, ...$ be a uniformly bounded sequence on $[a, b]$, and assume that $f_n \in \mathcal{R}[a, b]$. Let $F_n(x) = \displaystyle\int_a^x f_n(t)dt,\ x \in [a, b]$. Prove that there exists a subsequence $\{F_{n_k}\}$ of $\{F_n\}$ such that $\{F_{n_k}\}$ converges uniformly on $[a, b]$.

(b) Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on $[a, b]$. Call f *uniformly differentiable* on $[a, b]$ if, for every $\epsilon > 0$, there exists a $\delta > 0$, such that

$$\left|\frac{f(t) - f(x)}{t - x} - f'(x)\right| < \epsilon$$

for all $x, t \in [a, b]$ with $0 < |t - x| < \delta$. Prove that if f' is continuous on $[a, b]$ then f is uniformly differentiable on $[a, b]$.