MASTERS COMPREHENSIVE & PH.D. QUALIFYING EXAM
ANALYSIS: MATH 825/826 JANUARY 13, 2004, 2:00-5:00 P.M.

Instructions: *Answer 5 of the following 6 questions. Each question carries equal weight. If you work on more than five questions, clearly indicate which ones you want graded. Different parts of a question do not necessarily have the same weight. Use white paper and write on one side of the paper only. Please hand in your work in the correct order.*

(1) Prove the following limits exist and find their values:
a) $\lim_{n\to\infty} x_n$; where $x_1 = 1$ and $x_{n+1} = \sqrt{3 + x_n}$ for $n \in \mathbb{N}$.
b) $\lim_{n\to\infty} y_n$; where $y_n = \sum_{k=1}^{n} \frac{1}{n+k}$.

(2) This problem deals with the metric space $\mathbb{R}$.
a) Let $\{F_\lambda\}_{\lambda\in\Lambda}$ be a collection of closed subsets of $\mathbb{R}$ that satisfies the finite intersection property (i.e., the intersection of every finite sub-collection of $\{F_\lambda\}_{\lambda\in\Lambda}$ is nonempty). Further assume that for some $\lambda_0 \in \Lambda$, F_{λ_0} is bounded. Prove that: $\bigcap_{\lambda\in\Lambda} F_\lambda \neq \emptyset$.
(b) Find an open covering for the interval $(0, 1)$ that has no finite sub-cover. Prove your answer.
(c) Let $E \subseteq \mathbb{R}$. Do E and $\overline{E}$ (the closure of E) have the same interior? Prove your answer.

(3) Let $f_n(x) := \frac{x^2}{(x^2+1)^n}$, $x \in \mathbb{R}$. Carefully prove the following statements:
a) The series $\sum_{n=1}^{\infty} f_n(x)$ converges for every $x \in \mathbb{R}$; **but** not uniformly on $\mathbb{R}$.
b) $f_n \to 0$ uniformly on $\mathbb{R}$.

(4) Let $f : [a, b) \to \mathbb{R}$ be a given function.
a) Carefully state the Cauchy criterion for the existence of $f(b-) := \lim_{x\to b^-} f(x)$.
b) Assume that $f'(x)$ exists on $[a, b)$. Prove that: if $f'(x)$ is bounded on $[a, b)$, then $f(b-)$ exists.
c) Does the converse of part (b) hold? i.e., If $f'(x)$ exists on $[a, b)$ and $f(b-)$ exists, does it follow that $f'(x)$ is bounded on $[a, b)$?

(5) Let $f_n,: [a, b] \to \mathbb{R}$, $n = 1, 2, ...$ be given functions such that $\{f_n(a)\}$ is a convergent sequence, and $\{f_n''\}$ is a uniformly bounded sequence of continuous functions on $[a, b]$. Prove that there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ converges uniformly to some function f on $[a, b]$, f is continuously differentiable on $[a, b]$, and that $\{f'_{n_k}\}$ converges uniformly to f' on $[a, b]$. (See footnote)[1]

(6) a) Let $s \in (a, b)$. Let $f(x) = 0$ for all $x \neq s$ and $f(s) = 4$. Let α be an increasing function on $[a, b]$ with α is continuous at s. Prove that $f \in \mathcal{R}(\alpha)[a, b]$ (i.e., f is Riemann integrable with respect to α on $[a, b]$) and find $\int_a^b f d\alpha$.
b) Let

$$f(x) = \begin{cases} x; & x \in \mathbb{Q} \cap [0, 1] \\ 1; & x \in \mathbb{Q}^c \cap [0, 1] \end{cases}$$

Compute (with details and justification) the upper and lower Riemann integrals $\overline{\int_0^1} f(x)dx$ and $\underline{\int_0^1} f(x)dx$. Is f Riemann integrable on $[0, 1]$?

[1] question 5 is false as stated. One more assumption is needed. Namely, one has to assume $\{f'_n(a)\}$ is a bounded sequence. Without this additional assumption here is a trivial counter examples: take $f_n(x) = nx$, $x \in [0, 1]$. Thanks to Steve Haataja for pointing out the error and coming with the counter example.