MASTERS COMPREHENSIVE & PH.D. QUALIFYING EXAM
ANALYSIS JANUARY 21, 2005

Instructions: *Answer 5 of the following 6 questions. If you work on more than five questions, clearly indicate which ones you want graded. Each question carries equal weight. Different parts of a question do not necessarily have the same weight. Use white paper and write on one side of the paper only.*

(1) (a) For any $f : (0,1) \to (0,1)$, prove there is $\delta > 0$ with $\{x \in (0,1) : f(x) > \delta\}$ uncountable.

(b) Suppose that $\sum_{n=0}^{\infty} a_n$ converges and $a_n > 0$ for all n. Prove that $\sum_{n=0}^{\infty} (a_n a_{n+1})^{1/2}$ converges.

(2) (a) If (x_n) is a sequence of real numbers converging to x, prove $E = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact.

(b) If U is an open subset of $\mathbb{R}$, prove that U is countable union of compact subsets.

(3) (a) Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n + \sqrt{n}}$.

(b) Suppose $f_n : [0,1] \to \mathbb{R}$ are continuous functions converging uniformly to $f : [0,1] \to \mathbb{R}$. Either prove that $\lim_{n\to\infty} \int_{1/n}^{1} f_n(x)\, dx = \int_0^1 f(x)\, dx$ or give a counterexample.

(4) If $f : [0,1] \to \mathbb{R}$ is continuous, prove that $\lim_{n\to\infty} \int_0^1 f(x^n)\, dx = f(0)$.

(5) Suppose that (a_n) is a decreasing sequence of positive real numbers with $\sum_{n=1}^{\infty} a_n$ convergent. Prove that $\lim_{n\to\infty} n a_n = 0$.

(6) Suppose that $f : [0,1] \to \mathbb{R}$ is a differentiable function with $f(0) = 0$ and there is $K > 0$ so that $|f'(x)| \le K|f(x)|$ for all $x \in [0,1]$. Prove that $f(x) = 0$ for all $x \in [0,1]$.