Answer 5 of the following 7 questions. All have equal weight.

1. Let the power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ have radii of convergence R_1 and R_2 respectively.
   (a) If $R_1 \neq R_2$, prove that the radius of convergence, R, of the power series $\sum_{n=0}^{\infty}(a_n + b_n)x^n$ is $\min\{R_1, R_2\}$. What can be said about R when $R_1 = R_2$?
   (b) Prove that the radius of convergence, R, of the power series $\sum_{n=0}^{\infty} a_n b_n\, x^n$ satisfies $R \geq R_1 R_2$. Show by means of an example that the inequality can be strict.

2. (a) Give a careful $\epsilon - \delta$ proof that $g(x) = \sqrt{x}$ is continuous on $[0, \infty)$.
   (b) Assume that f is differentiable at a. Evaluate

$$\lim_{x \to a} \frac{a^n f(x) - x^n f(a)}{x - a} \qquad (n \in \mathbb{N})$$

3. (a) Let S be the set of all sequences $(q_1, q_2, \ldots)$ of rational numbers which converge to zero. Is S countable or uncountable?
   (b) Let (X, d) be a compact metric space and let $f : X \to X$ be continuous and onto. Let $g : X \to X$ and suppose $g \circ f$ is continuous. Prove that g must be continuous.

4. (a) Let

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n} \\ 0 & \text{otherwise} \end{cases}$$

   Show that $\sum_{n=1}^{\infty} f_n$ does not satisfy the Weierstrass M-Test but that it nevertheless converges uniformly on $\mathbb{R}$.

   (b) Suppose that f is continuous and $f(x) \geq 0$ on $[0, 1]$. If $f(0) > 0$, prove that $\displaystyle\int_0^1 f(x)\, dx > 0$.

5. Let $a_{m,n} \geq 0$ $(m, n \in \mathbb{N})$ and suppose that the partial sums

$$\sum_{m=1}^{M} \sum_{n=1}^{N} a_{mn}$$

   are bounded above. Prove carefully that $\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} a_{mn})$ and $\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} a_{mn})$ exist and are equal.

6. Suppose f' exists and is increasing on $(0, \infty)$ and that f is continuous on $[0, \infty)$ with $f(0) = 0$. Show that $g(x) = f(x)/x$ is increasing on $(0, \infty)$.

7. (a) Let f be continuous on $[0, 1]$ and $f(0) = f(1) = 0$. Show that there is a sequence of polynomials p_n such that $x(1 - x)p_n(x)$ converges to f uniformly.
   (b) Let f be uniformly continuous on a bounded set $E \subseteq \mathbb{R}$ and let $a \in \overline{E}$. Prove that $\lim_{x \to a} f(x)$ exists.