Analysis Qualifier, January 2007

Answer 5 of the following 6 questions. All questions are of equal weight. Note that in problems 2, 4 and 6, part (a) is independent of part (b).

1. Let $f(x) = \int_1^x \frac{1}{t}\, dt$ for $x > 0$.
   (a) Use an ϵ-δ proof to show that f is continuous on $(0, \infty)$.
   (b) Use an ϵ-δ proof to show that f is differentiable on $(0, \infty)$.

2. (a) Let $f : [1, +\infty) \to [0, +\infty)$ be piecewise continuous and nonincreasing. Show that the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ converges.
   (b) Let $g(t) = \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{t^j}{n^2}$.
      i. For what real values of t does this series converge absolutely?
      ii. For what real values of t does this series converge conditionally?

3. (a) Let B be a compact subset of $\mathbf{R}$ and let $f : B \to \mathbf{R}$ be continuous. Show that there exists a point $b \in B$ such that $f(x) \leq f(b)$ for all $x \in B$ (i.e. f attains its maximum value).
   (b) Let f be a positive continuous function defined on $\mathbf{R}$ such that $\lim_{|x| \to \infty} f(x) = 0$. Show that f attains its maximum value.

4. (a) Let $f : [0, 1] \to [0, 1]$ be defined by

$$f(t) = \begin{cases} 0 & x \text{ is irrational} \\ 1 & x = 0 \\ \frac{1}{q} & x = \frac{p}{q} \text{ in reduced form, where p and q are positive integers} \end{cases}.$$

   Prove that f is Riemann integrable on $[0, 1]$ and that $\int_0^1 f(x)\, dx = 0$.

   (b) For $n = 1, 2, 3, \ldots$ and $x \in [0, 1]$, let

$$g_n(x) = \frac{x^2}{x^2 + (1 - nx)^2}.$$

   Prove or disprove: The family of functions $\mathcal{G} = \{g_n : n = 1, 2, 3, \ldots\}$ is equicontinuous on $[0, 1]$.

5. Let E be a compact subset of $\mathbf{R}$.

   (a) Let $g : E \to \mathbf{R}$ be a continuous function such that $g(x) \neq 0$ for all $x \in E$. Show that there exists $c > 0$ such that $|g(x)| \geq c$ for all $x \in E$.

   (b) Suppose that a sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges uniformly on E to a *bounded* function f, and that a sequence of functions $\{g_n\}_{n=1}^{\infty}$ converges uniformly to a *continuous* function g, where $g(x) \neq 0$ for all $x \in E$. Prove that the sequence of functions $\{f_n/g_n\}_{n=1}^{\infty}$ is defined everywhere on E for large n and converges uniformly on E to f/g.

6. (a) Let f be a function of bounded variation on $[a, b]$. Furthermore, assume that for some $c > 0$, $|f(x)| \geq c$ on $[a, b]$. Show that $g(x) = 1/f(x)$ is of bounded variation on $[a, b]$.

   (b) Prove that every open set $A \subseteq \mathbf{R}$ can be written as a finite or countable union of *disjoint* open intervals (a_j, b_j), where at most one $a_j = -\infty$ and at most one $b_j = \infty$.