# Analysis Qualifier[1], January 2008

**Answer 5 of the following 6 questions.**

- If you work more than 5 problems, make sure that you clearly mark which problems you want to have counted.
- All problems are of equal weight, but the parts of a problem might not be of equal weight.
- The parts of a problem are not necessarily related.
- If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

1. (a) For the following series and intervals $\mathcal{J}$, determine (I) For what values of $x \in \mathcal{J}$ does the series converges, and (II) whether the series converges uniformly on the interval. Prove your answers.

   i. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{x}{n}\right)^2$, $\mathcal{J} = [-5, 5]$.

   ii. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{1+|x|^n}$, $\mathcal{J} = [-2, 2]$

   (b) Is the following true or false? If $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}$, then $f'(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$ for all $x \in \mathbb{R}$. Prove your answer, citing any relevant theorem(s).

2. Prove that if $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ is uniformly continuous, then there is a unique continuous function $g : \bar{A} \to \mathbb{R}$ such that $g|_A = f$. Here $\bar{A}$ is the closure of $A$ in $\mathbb{R}$.

3. Let $(X, d)$ be a complete metric space. A function $f : X \to X$ is called a *contraction* if there is a positive constant $\theta < 1$ such that

$$d(f(x), f(y)) \leq \theta d(x, y),$$

   for all $x, y \in X$.

   (a) Let $x_0 \in X$. Define the sequence $(x_n)$ recursively by $x_n = f(x_{n-1}), n = 1, 2, \ldots$. Prove that $(x_n)$ is a Cauchy sequence.

   (b) Show that $f$ has a fixed point, i.e. a point $z \in X$ such that $z = f(z)$.

   (c) Prove that the fixed point is unique.

   (d) Prove that there is a unique solution in the metric space $C[0, (1/2)]$ (with the sup norm) to

$$g'(x) = g(x), \qquad g(0) = 1.$$

   Hint: Write this problem as a fixed point problem $Tg = g$, with $(Tg)(x) = 1 + \int_0^x g(t)\, dt$.

[1]Problems 1(a)(ii) and 4(a) slightly revised.

4. (a) Let $q$ be a positive parameter and $x$ be a real variable. Consider the function

$$\theta_1(x) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+(1/2))^2} e^{(2n+1)x}.$$

Prove that if $0 < q < 1$, then the series converges absolutely for each $x \in \mathbb{R}$.

(b) Let $f_0 : [0,1] \to \mathbb{R}$, be absolutely integrable, and for $n = 1, 2, \ldots$ let

$$f_n(x) = \int_0^x f_{n-1}(t)\, dt.$$

Prove that $\lim_{n\to\infty} f_n = 0$ uniformly on $[0,1]$. Hint: First prove that $|f_1(x)| \leq M$ for some constant $M$.

5. (a) Prove or disprove: $h : [0,\pi] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is of bounded variation on $[0,\pi]$.

(b) i. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Prove, using the definition of the derivative:

$$\frac{d}{dx} \int_0^x f(t)\, dt = f(x).$$

ii. Find

$$\frac{d}{dx} \int_0^{x^2} \frac{1}{1+t^4}\, dt.$$

Make sure you identify any theorems you are using.

6. (a) Let $S$ denote the set of all rational numbers in the interval $[0,1]$. Prove, using the "open covering" definition of compactness, that $S$ is not a compact subset of $\mathbb{R}$.

(b) Suppose that $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$. Using an $\epsilon - \delta$ proof, show that if $t \neq s$ and $t_n \neq s_n$, then

$$\lim_{n\to\infty} \frac{s_n + t_n}{s_n - t_n} = \frac{s+t}{s-t}.$$

(c) For $n = 1, 2, \ldots$, set

$$b_n = \sum_{k=1}^{n} n^{-1} \sin\left(\frac{k\pi}{n}\right).$$

Either show that the sequence $(b_n)$ diverges, or find its limit as $n \to \infty$.