Masters & Ph.D. Qualifying Exam **Analysis: Math 825/826**

January 22, 2009, 2:00-6:00p.m., Avery Hall 347

• Work 5 complete question out of 6. • Each problem is worth 20 points.

• Write on one side of the paper only and hand your work in order. • Do not interpret a problem in such a way that it becomes trivial.

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(1) a) (10 points) Let $a_1 = 1$ and $a_{n+1} = \sqrt{a_n + 1}$ for $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} a_n$ exists, and find its value.

b) (10 points) For which real numbers x does the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n^x}$ converge? Justify your answer.

(2) Let $f : (0, 1] \longrightarrow \mathbb{R}$ be a uniformly continuous function on $(0, 1]$.

a) (10 points) State the definition of the uniform continuity of f on $(0, 1]$, and use it to show that $g(x) = \ln x$ is not uniformly continuous in $(0, 1]$.

b) (10 points) Prove that f can be uniquely extended to a continuous function on $[0, 1]$.

(3) (20 points) Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{R}$. Define A by:

$$A := \{a \in \mathbb{R} : \text{ the set } \{n \in \mathbb{N} : \ x_n < a\} \text{ is a finite set}\}.$$

Prove that: $\sup A = \liminf_{n\to\infty} x_n$.

(4) This problem has two independent parts.
a) (10 points) Consider the metric space $(\mathbb{Q}, d)$ where $\mathbb{Q}$ denotes the rational numbers and $d(x, y) = |x - y|$. Let $E := \{x \in \mathbb{Q} : \ x > 0, \ 2 < x^2 < 3\}$. Is E closed and bounded in $\mathbb{Q}$? Is E compact in $\mathbb{Q}$? Justify your answers.

b) (10 points) Let f be a continuous real-valued function on $[0, 1]$. Prove that there exists at least one point $\xi \in [0, 1]$ such that $\int_0^1 x^4 f(x)dx = \frac{1}{5} f(\xi)$.

(5) This problem has two independent parts.
a) (10 points) Let f be a function defined on $[0, 1]$ by: $f(x) = 0$ if x is irrational or $x = 0$, and $f(x) = 1/q$ if $x = p/q$ is rational where p, q have no common factor. Find the values of the upper and lower Riemann integrals, $\overline{\int_0^1} f dx$, $\underline{\int_0^1} f dx$. Is f Riemann integrable on $[0, 1]$?

b) (10 points) Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of real-valued, continuously differentiable functions on $[0, 1]$, such that, for all $n \in \mathbb{N}$,

$$|g_n'(x)| \le \frac{1}{\sqrt{x}}, \quad 0 < x \le 1; \text{ and } \int_0^1 g_n(x)dx = 0.$$

Prove that the sequence $\{g_n\}_{n=1}^{\infty}$ has a subsequence that converges uniformly on $[0, 1]$.

(6) This problem has two independent parts.
a) (10 points) Let $f_n(x) = \frac{nx}{1+n(1+x^2)}$, $n \in \mathbb{N}$. Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge uniformly on $\mathbb{R}$? and to what?

b) (10 points) Let g be a real-valued continuous function on $[0, 1]$. Evaluate the following limit: $\lim_{n\to\infty} n \int_0^1 x^n g(x)dx$. (Hint: First, consider: $\lim_{n\to\infty} n \int_0^1 x^n (g(x) - g(1)) dx$).