Analysis Qualifier Examination

Thursday, January 21, 2010, 2:00 – 6:00pm, Avery 347

• Work 5 out of 6 problems. • Each question is worth 20 points. • Write on one side of the paper only.

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1. (a) Determine whether or not the following sets are a) open, b) closed, and c) compact. (Try to find quick, easy justifications of each of the properties.)
   i. $\{(x, y) \in \mathbb{R}^2 : x + y < 3\}$
   ii. $\{z \in \mathbb{C} : |z| = 1\}$
   iii. $\{f \in C([-1, 1]) : f(0) = 0\}$ (using the metric $d(f, g) = \sup\{|f(x) - g(x)| : x \in [-1, 1]\}$)

   (b) Give an ϵ-δ proof that the function $f(x) = x^{1/3}$ is continuous at each point of $[0, 1]$.

2. Let S be a subset of the metric space (X, d). A point $x \in S$ is called a **condensation point** of S if for every $r > 0$, $B_r(x) \cap S$ is uncountable. Let $C(S)$ be the set of condenstation points of S.
   (a) Prove that the set $C(S)$ is closed.
   (b) Prove that if S is compact and uncountable, then $C(S)$ must be nonempty.
   (c) Prove that if (X, d) is the real line with the usual metric and S is uncountable, then $C(S)$ must be nonempty.

3. Suppose that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are sequences of nonzero real numbers such that for each n, $a_{n+6}/a_n \le b_{n+6}/b_n$.
   (a) Prove that if each a_n and b_n is positive, and $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.
   (b) Is the positivity of a_n and b_n necessary for the conclusion of part (a)? That is, if each a_n and b_n is a nonzero number such that for each n, $a_{n+6}/a_n \le b_{n+6}/b_n$ and $\sum_{n=0}^{\infty} b_n$ converges, must $\sum_{n=0}^{\infty} a_n$ converge? Either prove it or provide a counterexample.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$. If (x_n) is an increasing sequence of real numbers converging to a and (y_n) is an decreasing sequence of real numbers converging to a, prove that
$$\lim_{n\to\infty} \frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(a).$$

5. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, $f(x) \ge 0$ for every $x \in [a, b]$ and let $M = \sup\{f(x) : x \in [a, b]\}$. Put $N_p = \left(\int_a^b f(x)^p \, dx\right)^{1/p}$. Prove that $\lim_{p\to\infty} N_p = M$.

6. Determine whether or not the following series converge uniformly on the given intervals.
$$\text{a)} \sum_{n=1}^{\infty} x^2(1 - x^2)^{n-1} \text{ on } [-1, 1], \quad \text{b)} \sum_{n=1}^{\infty} \frac{(-1)^n \sin^n(x)}{n} \text{ on } [0, \pi], \quad \text{c)} \sum_{n=1}^{\infty} \frac{\ln(1 + nx)}{nx^n} \text{ on } [2, +\infty].$$