Analysis Qualifying Examination

Friday, January 21, 2011

INSTRUCTIONS: Work 5 of the following 6 problems. Write on only one side of each page. Each problem is worth 20 points.

1) The parts of this question are unrelated.

a) Define $f : [0, +\infty) \to \mathbb{R}$ by $f(x) = \sqrt{x}$. Using the definition of uniform continuity, prove that f is uniformly continuous on $[0, +\infty)$.

b) For $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, let $g(x, y) = \dfrac{xy^2}{x^2 + y^4}$. Prove that $\lim\limits_{(x,y)\to(0,0)} g(x, y)$ does not exist.

c) Suppose $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are convergent sequences of real numbers. Give a complete and careful proof of the following statement: if $a_n < b_n$ for every n, then $\lim\limits_{n\to\infty} a_n \le \lim\limits_{n\to\infty} b_n$.

2) Let a and b be real numbers with $a < b$.

a) Carefully prove that if $f : [a, b] \to \mathbb{R}$ is monotone increasing, then f is Riemann integrable on $[a, b]$.

b) Give an explicit example of a bounded function $g : [a, b] \to \mathbb{R}$ that is not Riemann integrable. (Be sure to prove that g is not integrable.)

3) Let (X, d) be a metric space, let $K \subseteq X$ be a compact subset of X, and let $F \subseteq X$ be a closed subset of X.

a) Suppose that $K \cap F = \emptyset$. Prove that there exists $\varepsilon > 0$ such that whenever $k \in K$ and $f \in F$, $d(k, f) \ge \varepsilon$.

b) For $x \in X$, define $\Delta(x, K) := \inf\{d(x, k) : k \in K\}$. (This quantity is often called the distance between x and K.) Suppose that $(x_n)_{n=1}^{\infty}$ is a sequence in X such that $\lim\limits_{n\to\infty} \Delta(x_n, K) = 0$, and let $S = \{x_n : n \in \mathbb{N}\}$. Prove that $S \cup K$ is compact.

4) Suppose that $f_n : [0, 1] \to \mathbb{R}$ are continuous functions converging uniformly to f, and that $g : \mathbb{R} \to \mathbb{R}$ is continuous.

a) Prove that the sequence $g \circ f_n$ converges uniformly to $g \circ f$.

b) Suppose that the domain of the functions f_n is $\mathbb{R}$ rather than $[0, 1]$ (but the hypotheses are otherwise unchanged). Does the result of the first part still hold? If so, provide a proof. If not, give a counterexample.

5) Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence of positive real numbers such that $\lim\limits_{n\to\infty} a_n = 0$ and the series $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Prove that for all $x > 0$ there exist integers $n(1) < n(2) < n(3) < \dots$ such that $\sum\limits_{k=1}^{\infty} a_{n(k)} = x$.

6) Suppose that $g : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function such that the partial derivative $\dfrac{\partial g}{\partial x}(x, y)$ exists and is continuous at every point $(x, y) \in \mathbb{R}^2$. Let $a < b$ be real numbers. A well-known theorem states that if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \displaystyle\int_a^b g(x, y)\, dy$, then f is differentiable and $f'(x) = \displaystyle\int_a^b \frac{\partial g}{\partial x}(x, y)\, dy$. Prove this theorem.