Analysis Qualifier Examination
Thursday, January 19, 2012.

- Work 5 out of 6 problems. • Each question is worth 20 points. • Write on one side of the paper only.

Question 1.

a) Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be bounded sequences of positive real numbers. Suppose that $\sum_{n=1}^{\infty} b_n$ is convergent. Show that $\sum_{n=1}^{\infty} a_n b_n$ is also convergent.

b) Let $y \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be given. Suppose for every sequence (x_n) we have $\liminf_{n\to\infty} |f(x_n) - f(y)| \le \liminf_{n\to\infty} |x_n - y|$. Prove that f is continuous at y.

Question 2.

a) Suppose $f : [0,\infty) \to \mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x) = L$. Find with proof $\lim_{n\to\infty} \int_0^1 f(nx)dx$.

b) Let $f, g : [a,b] \to \mathbb{R}$ be continuous and of bounded variation on $[a,b]$. Prove that $F(x) = f(x)g(x)$ must have a bounded variation on $[a,b]$.

Question 3.

a) Give an example of a metric space where every sequence with exactly one limit point is necessarily convergent. Prove your answer.

b) Let (X,d) be a metric space with metric d and suppose $x_0 \in X$. For $\varepsilon > 0$ define $E_\varepsilon := \{x \in X : d(x, x_0) \ge \varepsilon\}$. Suppose function $f : X \to \mathbb{R}$ is continuous and $f(E_\varepsilon)$ is compact in $\mathbb{R}$ for every $\varepsilon > 0$. Prove that $f(X)$ is compact.

Question 4.

a) Assume that $f : \mathbb{R} \to \mathbb{R}$ is Riemann-integrable and the improper integral $\int_{-\infty}^{\infty} |f(x)|dx$ converges. Prove that

$$F(x) = \int_0^x f(s)ds$$

is uniformly continuous on $\mathbb{R}$.

b) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f' \in C(\mathbb{R})$. Assume that there are $a, b \in \mathbb{R}$ such that $\lim_{x\to\infty} f(x) = a$ and that $\lim_{x\to\infty} f'(x) = b$. Prove that $b = 0$.

Question 5.

a) Let $f : (a,b) \to \mathbb{R}$ be differentiable and assume that $f'(c) > 0$ for some $c \in (a,b)$. Prove directly that f cannot have a local extremum at $x = c$.

b) The functions in a sequence $(f_n)_{n\in\mathbb{N}}$ are differentiable and $f_n(0) = 0$ for all $n \in \mathbb{N}$. Suppose that for each $n \in \mathbb{N}$, $|f_n'(x)| \le \dfrac{1}{1+(nx)^2}$ for all x. Prove that the series $\sum_{n=1}^{\infty} f_n^2(x)$ converges uniformly on $\mathbb{R}$.

Question 6.

Show that the improper integral $\displaystyle\int_0^{\infty} \frac{\sin x}{x} dx$ exists.