**Masters & Ph.D. Qualifying Exam** **Analysis: Math 825/826**

January 22, 2014, 12:00-6:00pm. Avery Hall 15

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• Work 5 out of 6 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.

• Do not interpret a problem in such a way that it becomes trivial.

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(1) The two parts of this problem are unrelated.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f'$ continuous on $\mathbb{R}$. Assume that there are $L, M \in \mathbb{R}$ such that $\lim_{x\to\infty} f(x) = L$ and that $\lim_{x\to\infty} f'(x) = M$. Prove that $M = 0$.

(b) Let $[a, b] \subset \mathbb{R}$ and $x_0 \in (a, b)$ be given. Set

$$P_0 := \{g : [a, b] \to \mathbb{R} \mid \ g \text{ is a polynomial satisfying } g(x_0) = 0\}.$$

Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and satisfies $f(x_0) = 0$. Must there be a sequence of polynomials in $P_0$ that converges uniformly to $f$ on $[a, b]$? (Justify your answer.)

(2) (a) Produce sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ of positive real numbers such that

$$\liminf_{n\to\infty}(a_n b_n) > (\liminf_{n\to\infty} a_n)(\liminf_{n\to\infty} b_n)$$

(b) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of positive real numbers. Suppose that $\{a_n\}_{n=1}^{\infty}$ is convergent. Prove that

$$\liminf_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\liminf_{n\to\infty} b_n).$$

(3) (a) Is there a function $f : \mathbb{R} \to \mathbb{R}$, that is differentiable on $\mathbb{R}$, such that

$$f'(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1, & \text{if } x \geq 1. \end{cases}$$

(Justify your answer.)

(b) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} x^2 \sin\left(\sin \frac{1}{x}\right) + \sin x + 1, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$$

Prove that $f$ is differentiable on $\mathbb{R}$ and that $f'$ is not continuous at 0.

(4) Let $(X, \rho)$ be a compact metric space, and suppose that $f : X \to X$ satisfies

$$\rho(f(x), f(y)) < \rho(x, y) \quad \text{for each } x, y \in X \text{ satisfying } x \neq y.$$

(a) Verify that $f$ is uniformly continuous on $X$.

(b) Prove that there exists a unique $x_0 \in X$ such that $f(x_0) = x_0$.

(5) Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a continuous derivative on $[-1, 1]$ and possesses a second derivative on $(-1, 1)$. Assume that $|f''(x)| \leq 1$ for each $x \in (-1, 1)$.

(a) Using Taylor's theorem, verify that

$$|f(x) + f(-x) - 2f(0)| \leq x^2$$

for each $x \in [-1, 1]$.

(b) Prove that

$$\left| \int_{-1}^{1} f(x)\, dx - 2f(0) \right| \leq \frac{1}{3}.$$

(c) Suppose also that $|f(x)| \leq 1$ for each $x \in [-1, 1]$. Show that $|f'(x)| \leq 2$ for all $x \in [-1, 1]$.

(6) Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of real numbers such that $\sum_{j=1}^{\infty} a_j$ is absolutely convergent. For each $n \in \mathbb{N}$ define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) := \sum_{j=1}^{n} a_j \cos(jx).$$

Prove that there is an $f : \mathbb{R} \to \mathbb{R}$ such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $f$ on $\mathbb{R}$.