Masters and Ph.D. Qualifying Exam **Analysis: Math 825/826**

Wednesday, January 21, 2015, 12:00 - 6:00p.m.

• Work 5 out of 6 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.

• Do not interpret a problem in such a way that it becomes trivial.

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(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. Fix $c \in \mathbb{R}$, and suppose that f has the following property: there is an L such that for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left| \frac{f(r) - f(c)}{r - c} - L \right| < \varepsilon \text{ whenever } r \in \mathbb{Q} \text{ and } 0 < |r - c| < \delta.$$

Prove that f is differentiable at c and that $f'(c) = L$.

(2) (a) Carefully state the Mean Value Theorem.

(b) Let $\lambda > 0$ be given. Show that there is no function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable at every $x \in \mathbb{R}$ and such that

$$f'(x) = \begin{cases} 0, & x < 0 \\ \lambda, & x \geq 0. \end{cases}$$

(3) Define $\alpha, \beta, f : \mathbb{R} \to \mathbb{R}$ by

$$\alpha(x) := \begin{cases} -1, & x < 0 \\ 4, & x \geq 0; \end{cases} \qquad \beta(x) := \begin{cases} -1, & x \leq 0 \\ 4, & x > 0; \end{cases} \qquad \text{and} \quad f(x) := \begin{cases} 0, & x < 0 \\ 1, & x \geq 0; \end{cases}$$

(a) Determine whether f is Riemann-Stieltjes integrable with respect to α over $[-1, 1]$. If it is evaluate $\int_{-1}^{1} f \mathrm{d}\alpha$.

(b) Determine whether f is Riemann-Stieltjes integrable with respect to β over $[-1, 1]$. If it is evaluate $\int_{-1}^{1} f \mathrm{d}\beta$.

(4) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of $\mathbb{R}$-valued functions defined on $\mathbb{R}$. Suppose that for each $n \in \mathbb{N}$ and each $x \in \mathbb{R}$ we have $0 \leq f_{n+1}(x) \leq f_n(x)$ and that $f_n \to 0$ uniformly on $\mathbb{R}$. Prove that $\sum_{n=1}^{\infty} (-1)^n f_n(x)$ converges uniformly on $\mathbb{R}$.

(5) Let (X, ρ) and (Y, σ) be metric spaces.

(a) Carefully state the definition of continuity for a mapping $f : X \to Y$.

(b) Carefully state the definition of compactness for X.

(c) Suppose that X is compact and that $f : X \to Y$ is continuous. Prove that $f(X)$ must be a compact subset of Y.

(6) Suppose that $\{a_n\}_{n=1}^{\infty}$ is a convergent sequence of real numbers. Suppose that $M \in \mathbb{R}$ is such that both $\lim_{n\to\infty} a_n \neq M$ and $a_n \neq M$ for all $n \in \mathbb{N}$. Show that there must be a $d > 0$ such that $|a_n - M| > d$ for all $n \in \mathbb{N}$.