- Answer **five** out of the following seven questions.
- If you answer more than five, make clear which questions you want graded.
- All questions carry equal weight.

---

1. Suppose $g_n(x)$ is a sequence of real-valued functions defined on a set $S \subseteq \mathbb{R}$. Suppose further that $0 \leq g_{n+1}(x) \leq g_n(x)$ for all $n \in \mathbb{N}$ and $x \in S$ and that $g_n(x) \to 0$ uniformly on S. Prove that $\sum_{n=1}^{\infty}(-1)^n g_n(x)$ converges uniformly on S.

2. (a) Let S be a set of real numbers. Prove carefully from the definitions that

$$\sup\{|x-y| \ : \ x, y \in S\} = \sup(S) - \inf(S)$$

   (b) Hence prove that whenever f is a Riemann integrable function on $[a,b]$ then $|f|$ is also Riemann integrable on $[a,b]$.

3. Let $f(x,y)$ be continuous on $[a,b] \times [c,d]$. Prove that the function

$$g(x) := \int_c^d f(x,y)\,dy$$

   is continuous on $[a,b]$.

4. (a) Define the term
   it compact set in a metric space.

   (b) Let (X, ρ) be a metric space and let (x_n) be a sequence in X that converges to $a \in X$. Prove directly from the definition of compactness that

$$K := \{a\} \cup \{x_n \ : \ n \in \mathbb{N}\}$$

   is compact.

5. Let $h : \mathbb{R} \to \mathbb{R}$. Show that $\lim_{t \to a+} h(t)$ exists if and only if, given any $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $a < x, y < a + \delta$.

6. Suppose that $f : [0, +\infty) \to \mathbb{R}$ and that

$$f(0) = 0, f'(0) = 1 \text{ and } f''(x) \leq 0 \text{ for all } x > 0$$

   (a) Prove that $f(x) \leq x$ for all $x \geq 0$.

   (b) Prove that $f(x)/x$ is decreasing on $[0, +\infty)$.

7. Suppose that a subset S of a metric space has the property that given any $a, b \in S$, there is a continuous function $\gamma : [0,1] \to S$ such that $\gamma(0) = a$ and $\gamma(1) = b$. Prove that S is connected.