**June 1999**
**825/826 Qualifying Exam**

**I. Do 3 of 5.**

1. Let $A$ be a countable set. Consider the power set of $A$, $\mathcal{A} = \{S : S \subset A\}$. Is $\mathcal{A}$ countable or uncountable? Prove your answer.

2. Consider the set $B = \{1 - \frac{1}{n} : n = 1, 2, 3, \ldots\}$.
   1. Is $B$ open? closed? Prove your answers. Provide all details.
   2. Is $B$ compact? Prove your answer directly from the definition and by stating a theorem.

3. Let $\alpha$ be a monotonically increasing function on $[a, b]$ and let $f$ be a real function bounded on $[a, b]$.
   a. Define the Riemann-Stieltjes integral of $f$ with respect to $\alpha$ over $[a, b]$.
   b. Suppose
   $$\alpha(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 3 & x > 0 \end{cases}$$
   Under what conditions is $f \in \Re(\alpha)$ on $[-2, 2]$? Prove your answer. (there is only one direction to prove here)
   c. If $f \in \Re(\alpha)$, compute $\int_{-2}^{2} f d\alpha$. Justify your answer.

4. Consider the $2\pi$-periodic function given by $f(x) = |x|$ for $-\pi \leq x \leq \pi$ and $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$. (You'll need $\int xe^{ax}\, dx = xe^{ax}/a - e^{ax}/a^2$)
   a. Compute the Fourier series for $f$ and comment on the covergence of the series.
   b. Compute the sums
   $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$
   and
   $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$$

5a. Suppose $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$. Give a formula relating the total and partial derivatives of $f$ and state conditions under which the formula holds.
b. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable in an open set $E \subset \mathbb{R}^n$ and that $f$ has a local maximum at a point $\mathbf{x} \in E$. Prove that $f'(\mathbf{x}) = 0$.

**II. Do 4 of 5.**

6. If $f$ is a continuous mapping of a metric space $X$ into a metric space $Y$, prove that
   $$f(\bar{E}) \subset \overline{f(E)}$$
   for all $E \subset X$.

7. Let $X$ be a metric space and $E \subset X$.
   a. What does it mean to say $E$ *is open*? $E$ *is closed*? $x$ *is a limit point (or accumulation point) of* $E$?
   b. Let $F = E'$ be the set of all limit points of $E$. Prove that $F$ is closed.

c. Is $E' = (E')'$? Justify your answer.

8a. Consider a sequence $\{x_n\} \subset \mathbb{R}$. Suppose $x_{2n} \leq x_{2n+2} \leq x_{2n+1} \leq x_{2n-1}$ for all $n$ (i.e. the even terms of the seqeunce are increasing and the odd terms are decreasing and all of the even terms are smaller than all of the odd terms). Also suppose $\lim_{n\to\infty}(x_{2n-1} - x_{2n}) = 0$. Prove there exists an $x \in \mathbb{R}$ such that $x_n \to x$ and, for all $n$, $x_{2n} \leq x \leq x_{2n-1}$.

b. Consider the sequence given by

$$x_1 = 1$$

$$x_{n+1} = \frac{1}{1 + x_n}$$

Show that this sequence converges and compute its limit. (Hint: to prove $\lim_{n\to\infty}(x_{2n-1} - x_{2n}) = 0$, use induction to show $x_{2n-1} - x_{2n} < \left(\frac{5}{9}\right)^n$).

9. Let $f$ be a real uniformly continuous function on the bounded set $E \subset \mathbb{R}$. Prove that $f$ is bounded on $E$. Show that the conclusion may be false if boundedness of $E$ is not assumed.

10a. Prove that if $f$ is differentiable in $(a, b)$ and if $f'(x) = 0$ for all $x \in (a, b)$, then $f$ is constant on $(a, b)$.

b. Let $f$ be defined for all $x \in \mathbb{R}$ and suppose that for all $x, y$ we have $|f(x) - f(y)| \leq (x - y)^2$. Prove $f$ is constant.