Answer 6 of the following 8 questions. All have equal weight.

1. (a) Define (a_n) by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. Prove that (a_n) converges, find its limit, and prove your answer is correct.

   (b) Let (x_n) be a bounded sequence in $\mathbb{R}$. Prove that if every convergent subsequence of (x_n) converges to the same limit, then $\lim x_n$ exists.

2. Determine whether the following series converge or diverge, proving your answer.

$$\sum_{n=1}^{\infty} \sqrt{n^2+1} - n, \qquad \sum_{n=1}^{\infty} (n^{1/n} - 1)^n.$$

3. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x \operatorname{sgn}\left(\sin \frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

   where $\operatorname{sgn}(x)$ is 1 if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$. Answer the following questions, proving your answer in each case.

   (a) Is f continuous at 0 ?

   (b) Is f differentiable at 0 ?

   (c) Is f Riemann integrable on $[-1, 1]$?

4. Let $x > 0$ and $\alpha \in \mathbb{R}$. Show that $(1 + x)^\alpha > 1 + \alpha x$ if $\alpha < 0$ or $\alpha > 1$ and that $(1 + x)^\alpha < 1 + \alpha x$ if $0 < \alpha < 1$.

5. Prove the following theorem: If, for each $n \in \mathbb{N}$, $g_n : [a, b] \to \mathbb{R}$ is a continuous nonnegative function and $\sum_{n=1}^{\infty} g_n$ converges pointwise to a continuous function, then $\sum_{n=1}^{\infty} g_n$ converges uniformly on $[a, b]$.

6. Suppose that (X, ρ) and (Y, σ) are metric spaces, X is compact, and $f : X \to Y$ is continuous.

   (a) Show that $f(X)$ is compact.

   (b) Define uniform continuity of f.

   (c) Show that f is uniformly continuous.

7. Prove that the set of polynomials with rational coefficients is dense in $C([0,1])$, the vector space of continuous functions $f : [0,1] \to \mathbb{R}$ with the norm $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$.

8. (a) Consider two functions $g, h : \mathbb{R} \to \mathbb{R}$, where g and h are differentiable at $a \in \mathbb{R}$. From the definition, show that the function $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x,y) = (g(x), h(y))$ is differentiable at a.

   (b) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $D_1 f = \dfrac{\partial f}{\partial x_1}$ exists and is identically zero. Show that there is $g : \mathbb{R}^{n-1} \to \mathbb{R}$ so that $f(x_1, x_2, \ldots, x_n) = g(x_2, \ldots, x_n)$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$.