MASTERS COMPREHENSIVE & PH.D. QUALIFYING EXAM
ANALYSIS: MATH 825/826 JUNE 3, 2002

Instructions: *Answer 5 of the following 7 questions. All have equal weight.*

(1) Let $f : (0, 1) \to \mathbb{R}$ be a function.
(a) Give the technical definition (the $\epsilon - \delta$ definition) of $\lim_{x\to a} f(x) = L$ where $a \in (0, 1)$ and $L \in \mathbb{R}$.
(b) Use the definition only to show that $\lim_{x\to a} \frac{x^2}{1-x} = \frac{a^2}{1-a}$ for every $a \in (0, 1)$.

(2) Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ be sequences in a metric space (X, d) with $x_n \to x_0$ and $y_n \to y_0$ as $n \to \infty$.
(a) Prove that $\lim_{n\to\infty} d(x_n, y_n) = d(x_0, y_0)$.
(b) Prove that the set $E = \{x_n : n = 0, 1, 2, 3, ...\}$ is compact in X.

(3) Let $f : [a, b) \to \mathbb{R}$ be a function with $f'(x)$ exists on $[a, b)$.
(a) Prove that: If $f'(x)$ is bounded on $[a, b)$, then $f(b-) = \lim_{x\to b^-} f(x)$ exists. In this case, prove that f is uniformly continuous on $[a, b)$.
(b) Does the converse of part (a) hold? i.e., If $f'(x)$ exists on $[a, b)$ and $f(b-)$ exists, does it follow that $f'(x)$ is bounded on $[a, b)$?

(4) (a) Prove that $g_n(x) = \frac{nx}{1+n^2x^2} \to 0$ point-wise on $(0, \infty)$; **but not uniformly.**
(b) Let $\sigma > \frac{1}{2}$ and $f_n(x) = \frac{x}{n^\sigma(1+nx^2)}$, $x \in \mathbb{R}$, $n = 1, 2, 3, ...$

Prove that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $\mathbb{R}$.

(5) Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function on $[0, \infty)$.
(a) Prove that: If $\int_0^x f(t)dt = \int_x^1 f(x)dx$ for all $x \in [0, 1]$ then $f(x) = 0$ on $[0, 1]$.
(b) Prove that: If $f'(4) = 1$ then the sequence of functions $\{f_n(x) = f(nx) : n \in \mathbb{N}\}$ cannot be equicontinuous on $[0, \infty)$

(6) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function on $[0, 1]$.
(a) Prove that: If $\int_0^1 f(x)^2dx = 0$ then $f(x) = 0$ for all $x \in [0, 1]$.
(b) Prove that: If $\int_0^1 x^{3n}f(x)dx = 0$ for all $n = 0, 1, 2, ...$, then $f(x) = 0$ for all $x \in [0, 1]$.

(7) Let $f : [-1, 1] \to \mathbb{R}$ be a bounded function on $[-1, 1]$. Let α be given by: $\alpha(x) = 0$ if $x \le 0$, and $\alpha(x) = 1$ if $x > 0$. Prove that: $f \in \mathcal{R}(\alpha)[-1, 1]$ (i.e., f is Riemann integrable with respect to α on $[-1, 1]$) if and only if $f(0+) = f(0)$, where $f(0+) = \lim_{x\to 0^+} f(x)$. In this case, show that $\int_{-1}^{1} f d\alpha = f(0)$.