MASTERS COMPREHENSIVE & PH.D. QUALIFYING EXAM
ANALYSIS: MATH 825/826 JUNE 2, 2003, 1:00-4:00 P.M.

Instructions: *Answer 5 of the following 6 questions. Each question carries equal weight. If you work on more than five questions, clearly indicate which ones you want graded. Different parts of a question do not necessarily have the same weight. Use white paper and write on one side of the paper only.*

(1) Let (X, d) be a metric space and K be a subset of X.
(a) Give a careful definition of what it means to say that K is a compact subset of X. Also, define what an open covering of a subset of X is.
(b) Show by example that the union of infinitely many compact subsets of a metric space need not be compact.
(c) Let K be a compact subset of X and $x_0 \notin K$. The distance between x_0 and K is defined by: $d(x_0, K) = \inf \{d(x_0, y) : y \in K\}$. Prove that there exists a point $y_0 \in K$ such that $d(x_0, K) = d(x_0, y_0)$.

(2) (a) Let S be a collection of nonempty, open, mutually disjoint subsets of $\mathbb{R}$, i.e., if $A, B \in S$ then $A \neq \emptyset$, $B \neq \emptyset$; both are open in $\mathbb{R}$; and $A \cap B = \emptyset$. Prove that S is at most countable.
(b) Let $f : [a, b] \to [0, \infty)$ be a function with the property that for every choice of a finite number of distinct points $x_1, x_2, ..., x_n \in [a, b]$, the sum: $\sum_{i=1}^{n} f(x_i) \leq 1$.
Show that the set $S := \{x \in [a, b] : f(x) > 0\}$ is at most countable.

(3) Let f be a real-valued function such that f, f', and f'' are all continuous on $[0, 1]$. Consider the series $\sum_{k=1}^{\infty} f(\frac{1}{k})$.
(a) Prove that if the series $\sum_{k=1}^{\infty} f(\frac{1}{k})$ is convergent, then $f(0) = 0$ and $f'(0) = 0$.
(b) Conversely, show that if $f(0) = f'(0) = 0$, then the series $\sum_{k=1}^{\infty} f(\frac{1}{k})$ is convergent.

(4) Let $f_n(x) := nxe^{-nx}$, $x \in [0, \infty)$ and $n = 1, 2, ...$
(a) Find (with a proof) the point-wise limit $f(x) := \lim_{n\to\infty} f_n(x)$ for $0 \leq x < \infty$, and show that $\{f_n\}$ does not converge uniformly to f on $[0, \infty)$.
(b) Prove that $f_n \to f$ uniformly on $[\delta, \infty)$ for any $\delta > 0$.

(5) Let f be positive, continuous function on $[a, b]$; and let $M := \sup\{f(x) : x \in [a, b]\}$.
Show that $M = \lim_{n\to\infty} \left(\int_a^b (f(x))^n dx \right)^{\frac{1}{n}}$.

(6) (a) Let f be nonnegative, continuous function on $[a,b]$; and $\alpha : [a,b] \to \mathbb{R}$ is strictly increasing on $[a,b]$. Show that, if $\int_a^b f d\alpha = 0$, then $f \equiv 0$ on $[a,b]$.

(b) Let α be given by:

$$\alpha(x) = \begin{cases} 0; & 0 \leq x < 1 \\ 2; & 1 \leq x < e \\ 5; & e \leq x \leq \pi \end{cases}$$

Either directly or by the aid of a theorem, calculate the value of the integral $\int_0^\pi x^{100} d\alpha$, and show all the details in your calculation.