MASTERS COMPREHENSIVE & PH.D. QUALIFYING EXAM
ANALYSIS: MATH 825/826 JUNE 4, 2004

Instructions: *Answer 5 of the following 6 questions. If you work on more than five questions, clearly indicate which ones you want graded. Each question carries equal weight. Different parts of a question do not necessarily have the same weight. Use white paper and write on one side of the paper only.*

(1) Let (a_n) and (b_n) be bounded sequences of real numbers.
  (a) Prove that $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.
  (b) Show that if (a_n) converges, then $\limsup_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.
  (c) Prove that if, for a fixed (a_n) and for all (b_n), $\limsup_{n\to\infty}(a_n+b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$, then (a_n) converges.

(2) Suppose that $f : [a, b] \to \mathbb{R}$ has a continuous derivative on $[a, b]$. Prove that f is uniformly differentiable, in the sense that for any $\epsilon > 0$, there is $\delta > 0$ so that for all $x, y \in [a, b]$ with $0 < |x - y| < \delta$,

$$\left| f'(x) - \frac{f(x) - f(y)}{x - y} \right| < \epsilon.$$

(3) Define $H : \mathbb{R} \to \mathbb{R}$ by $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x \geq 0$. Let $f : [0, 1] \to \mathbb{R}$ be continuous.
  (a) Suppose that $0 < t < 1$. Show that if $g(x) = H(x - t)$, then $\int_0^1 f\,dg = f(t)$.
  (b) Suppose that (t_n) is a sequence of distinct points in $(0, 1)$ and (c_n) is a sequence of positive numbers with $\sum_{n=1}^{\infty} c_n < \infty$. Show that if $g(x) = \sum_{n=1}^{\infty} c_n H(x - t_n)$, then g is well-defined and

$$\int_0^1 f\,dg = \sum_{n=1}^{\infty} c_n f(t_n).$$

(4) (a) Consider the functions $f_n(x) = \dfrac{n^2 x}{1 + n^3 x^2}$ for $x \geq 0$. Show that f_n converge pointwise to zero on $[0, \infty)$. For which $a \geq 0$, if any, does f_n converge uniformly on $[a, +\infty)$?
  (b) Suppose that a sequence of uniformly continuous functions $f_n : [0, +\infty) \to \mathbb{R}$ converge uniformly to a function $f : [0, +\infty) \to \mathbb{R}$. Prove that f is uniformly continuous.

(5) (a) Suppose A and B are distinct points in $\mathbb{R}^2$. Prove that there is a circle in $\mathbb{R}^2$ that passes through A and B but does **not** pass through any point in $\mathbb{Q}^2 \setminus \{A, B\}$.
  HINT: First find an uncountable family of circles passing through A and B.
  (b) Determine convergence or divergence of these series: (i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log n}$, (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$.

(6) Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous, differentiable on $(0, 1)$, $f(0) = f(1) = 0$ and there is $x \in (0, 1)$ with $f(x) = 1$. Prove there is some $c \in (0, 1)$ with $|f'(c)| > 2$.