# Real Analysis Qualifying Examination–Math 825/826

June 1, 2005, 1:00-5:00 p.m., Avery Hall 19

• **Work 5 out of 6 problems.** • Each problem is worth 20 points, but different parts do not necessarily have the same weight. • Write on one side of the paper only and hand your work in order. • Write your exam ID Code on every paper.

• $\mathbb{R}$ denotes the real numbers, and $\mathbb{Q}$ denotes the rational numbers.

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(1) (a) Use the definition of derivative to prove that if $f$ and $g$ are differentiable at a point $x$, then $f \cdot g$ is differentiable at $x$.

(b) Use the definition of the Riemann integral to prove that if $f$ is bounded on $[a, b]$ and is continuous everywhere except for finitely many points in $(a, b)$, then $f$ is Riemann integrable on $[a, b]$.
(*Note:* no points will be given for citing direct references to both results.)

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(2) (a) Give an $\epsilon$-$\delta$ proof that the function $f(x) = \dfrac{x+2}{x^2 - x}$ is continuous on the open interval $(0, 1)$.

(b) Does the improper Riemann integral $\displaystyle\int_0^1 \frac{\sin x}{x^{3/2}} dx$ exist? Prove your assertion.

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(3) (a) Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n$ **converges conditionally**. Let $p_1, p_2, \ldots$ denote the positive terms of $\sum_{n=1}^{\infty} a_n$ in the same order in which they occur; similarly let $q_1, q_2, \ldots$ denote the negative terms of $\sum_{n=1}^{\infty} a_n$ in the same order in which they occur. Use the definition of convergent series to show that both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ diverge.

(b) If the series $\sum_{n=0}^{\infty} a_n$ converges conditionally, show that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is 1.

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(4) (a) Prove or disprove: The sequence of functions $f_n(x) = x(1-x)^n$ converges uniformly on $[0, 1]$ as $n \to \infty$.

(b) Let $g_n(x) = x^n(1 - x^n)$ and $\mathbb{F} := \{g_n : n = 1, 2, \ldots\}$. Is $\mathbb{F}$ equicontinuous on $[0, 1]$? Prove your assertion.

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(5) (a) Suppose $f \in C[0, 1]$ and $\displaystyle\int_0^1 f(x)x^n dx = 0$ for all $n = 99, 100, 101, \ldots$. Show that $f \equiv 0$.

(b) Let $S \subset \mathbb{R}$ be uncountable. Show that there is an $x_0 \in \mathbb{R}$ such that every neighborhood of $x_0$ has uncountably many points of $S$.

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(6) (a) A function is of bounded variation on $[a, b]$ if there is a number $K$ such that for every partition $a = a_0 < a_1 < \cdots < a_n = b$ of $[a, b]$, $\displaystyle\sum_{j=1}^{n} |f(a_j) - f(a_{j-1})| \le K$. Let

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q}. \end{cases}$$

Show that $f$ is NOT of bounded variation on $[0, 1]$.

(b) Let

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or if } x \in [0,1] \setminus \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in [0,1] \cap \mathbb{Q} \text{ in lowest terms.} \end{cases}$$

(To say that $x = \frac{p}{q}$ in lowest terms means that $p$ and $q$ are positive integers with no common factors.) Show that $g$ is continuous at every irrational of $[0, 1]$.

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**The End**