Math 825-826 Qualifier, June 2007
Answer any five of the following six questions. All questions are of equal weight.

(i) If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

(ii) Please note that parts (a) and (b) of any given problem are not necessarily related.

1. (a) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Prove that

$$\lim_{n\to\infty} \sup(-a_n) = -\lim_{n\to\infty} \inf a_n,$$

showing all details.

(b) Given any real number $\alpha \geq 0$, prove that it has a square root.

2. (a) Give a rigorous proof that

$$\lim_{x\to\infty} \frac{x^2+x}{2x^2+1} = \frac{1}{2}.$$

(b) Define the vector space $\mathcal{D}(0,1)$ as follows:

$$\mathcal{D}(0,1) \equiv \{\phi \in C^{\infty}([0,1]) : \phi(0) = \phi(1) = 0\}.$$

Show that if $f \in C([0,1])$ has the property that

$$\int_0^1 f(x)\phi(x)dx = 0 \text{ for every } \phi \in \mathcal{D}(0,1),$$

then f must be identically zero.

3. (a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are differentiable, that $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, and that $f(x_0) = g(x_0)$ for some $x_0 \in \mathbb{R}$. Prove that $f'(x_0) = g'(x_0)$.

(b) Let f be any function in $C^1([a,b])$ which satisfies $f(a) = 0$. Show that there exists a positive constant M, independent of f, such that one has the inequality

$$\int_a^b |f(x)|^2\, dx \leq M \int_a^b |f'(x)|^2\, dx.$$

4. (a) Let

$$f_n(x) = \frac{x}{x+n}; \;\; g_n(x) = \frac{nx}{1+n^2x^2}.$$

For each sequence $\{f_n\}$ and $\{g_n\}$, determine whether one has: (i) pointwise convergence on $(0,1)$; (ii) uniform convergence on $(0,1)$.

(b) Evaluate each of the following:

$$(i) \quad \lim_{n\to\infty} \int_{\frac{\pi}{2}}^{\pi} \frac{n \sin(x/n)}{x} dx; \quad (ii) \quad \sum_{n=1}^{\infty} n^2 x^n, \;\; |x| < 1$$

(Justify your reasoning for each of these.)

5. (a) i. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$. Prove that f is uniformly continuous on any finite interval $[a,b] \subset \mathbb{R}$.

ii. A function f is said to be *periodic* if there exists $p > 0$ such that $f(x+p) = f(x)$, for all $x \in \mathbb{R}$. Prove that a continuous and periodic function on $\mathbb{R}$ is uniformly continuous on $\mathbb{R}$

(b) Give an example of a function that is uniformly continuous on $\mathbb{R}$, but not Lipschitz continuous. (Of course, justify your reasoning here.)

6. (a) Let $\alpha(t) := n^2$ for $t \in [n, n+1)$. Calculate the Riemann-Stieltjes integral $\int_0^4 x^2\, dx$ from the definition.

(b) Let

$$f(x) = \begin{cases} x^2(1-x) & x \text{ rational} \\ 0 & x \text{ irrational.} \end{cases}$$

Show that f is continuous at $x = 0$ and $x = 1$, that $f'(0)$ exists, and that $f'(1)$ does not exist.