Masters & Ph.D. Qualifying Exam **Analysis: Math 825/826**

Monday, June 2, 2008, 2:00-6:00p.m., Avery Hall 110

• Work 5 out of 6 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.

• Do not interpret a problem in such a way that it becomes trivial.

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(1) Let $f(x) = \dfrac{x}{1-x}$, $x \in (0,1)$.

a) (10 points) By using the ϵ–δ definition of the limit only, prove that $\lim_{x\to t} f(x) = f(t)$, for every $t \in (0,1)$, i.e., f is continuous on $(0,1)$. (Note: You are not allowed to trivialize the problem by using properties of limits).
b) (10 points) Is f uniformly continuous on $(0,1)$? Justify your answer.

(2) Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in a metric space (X,d). Assume that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} y_n = y_0$ for some x_0, $y_0 \in X$.

a) (10 points) Prove that $\lim_{n\to\infty} d(x_n, y_n) = d(x_0, y_0)$.
b) (10 points) Is the set $E := \{x_n : n = 0, 1, 2, \cdots\}$ compact in X? Prove your answer.

(3) Let $f : (a,b] \longrightarrow \mathbb{R}$ be a function.

a) (14 points) Prove the following: If f' exists and is bounded on $(a,b]$, then $\lim_{x\to a^+} f(x)$ exists.
b) (6 points) Prove or disprove: If f' exists on $(a,b]$ and $\lim_{x\to a^+} f(x)$ exists, then f' is bounded on $(a,b]$.

(4) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers.

a) (10 points) Prove that: $\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$; whenever the right hand side of the inequality is not of the form $\infty - \infty$.
b) (10 points) Assume $\beta > 0$, $a_n > 0, n = 1, 2, 3, \cdots$; and the series $\sum_{n=1}^{\infty} a_n$ is divergent. Prove that the series $\sum_{n=1}^{\infty} \frac{a_n}{\beta + a_n}$ is divergent.

(5) Let $f_n(x) = \frac{x}{1+nx^2}$, for $n \in \mathbb{N}$. Let $C[a,b]$ denote the metric space of real-valued continuous function on $[a,b]$ with metric ρ defined by: $\rho(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|$ for $f, g \in C[a,b]$. Let $\mathcal{F} := \{f_n : n = 1, 2, 3, \cdots\}$ and $[a,b]$ be any compact segment in $\mathbb{R}$.

a) (6 points) Is $\mathcal{F}$ equicontinuous on $[a,b]$? Justify your answer.
b) (6 points) Is $\mathcal{F}$ compact in the metric space $(C[a,b], \rho)$? Justify your answer.
c) (8 points) Let $p > 0$ be fixed. Prove that, for any $a > 0$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} f_n(x)$ converges uniformly on $[a, \infty)$.

(6) Let $\alpha(x) = 0$ if $x \le 0$; $\alpha(x) = 1$ if $x > 0$, and $\beta(x) = x$. Assume that f is a bounded real-valued function on $[-1,1]$.

a) (10 points) Prove the following: $f \in \mathcal{R}(\alpha)$ on $[-1,1]$ if and only if $\lim_{x\to 0^+} f(x) = f(0)$. In the case $f \in \mathcal{R}(\alpha)$ on $[-1,1]$, find the value of $\int_{-1}^{1} f(x)d\alpha(x)$. (Note: $f \in \mathcal{R}(\alpha)$ on $[-1,1]$ means that f is Riemann integrable with respect to α on $[-1,1]$).
b) (10 points) Assume that the function $f \in \mathcal{R}(\beta)$ on $[-1,1]$, $m \le f(x) \le M$ for $x \in [-1,1]$, ϕ is continuous on $[m,M]$, and $g(x) := \phi(f(x))$. Prove that $g \in \mathcal{R}(\beta)$ on $[-1,1]$.