Masters & Ph.D. Qualifying Exam **Analysis: Math 825/826**

June 1, 2009, 2:00-6:00p.m., Avery Hall 110

• Work 5 complete question out of 6. • Each problem is worth 20 points.

• Write on one side of the paper only and hand your work in order. • Do not interpret a problem in such a way that it becomes trivial.

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(1) Let $f(x) = \dfrac{x^2}{1-x^2}$, $x \in (0,1)$.
a) By using the ϵ-δ definition of the limit only, prove that f is continuous on $(0,1)$.
b) Is f uniformly continuous on $(0,1)$? Prove your answer.

(2) Let (X,d) and (Y,ρ) be metric spaces.
a) Prove: If (X,d) is an unbounded and connected metric space, then for every $x_0 \in X$ and every $r > 0$ the set $\{x \in X : d(x,x_0) = r\}$ is nonempty.
b) Let $f,\ g : X \to Y$ be continuous functions, and E is dense subset of X. Prove that $f(E)$ is dense in $Y_1 = f(X)$; and if $g(x) = f(x)$ for all $x \in E$, then $g(x) = f(x)$ for all $x \in X$.

(3) a) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of of positive real numbers. Prove that $\sum_{n=1}^{\infty} a_n < \infty$ implies that $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}} < \infty$ **and** that the converse is false.
b) Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be bounded sequences in $\mathbb{R}$. Prove that

$$\liminf_{n\to\infty}(x_n + y_n) \geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n,$$

and show that the inequality can be strict.

(4) a) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued continuous functions such that $f_n \to f$ uniformly on $[0,1]$. Prove that $\lim_{n\to\infty} f_n(x_n) = f(x)$; whenever x_n, $x \in [0,1]$ satisfying $x_n \to x$, as $n \to \infty$.
b) Prove that the series: $x^2 + \dfrac{x^2}{1+x^2} + \dfrac{x^2}{(1+x^2)^2} + \dfrac{x^2}{(1+x^2)^3} + \dfrac{x^2}{(1+x^2)^4} + \cdots$ converges uniformly on $[a,\infty)$ for every $a > 0$; but not uniformly on $[0,b]$ for any $b > 0$.

(5) a) Let f be a bounded real-valued function on $[-1,1]$ and $\alpha(x) = 0$ if $x \leq 0$, $\alpha(x) = 1$ if $x > 0$. Prove that:
$f \in \mathcal{R}(\alpha)[-1,1]$ (i.e., f is Riemann integrable with respect to α on $[-1,1]$) if and only if f is right continuous at $x = 0$.
b) Let ϕ be a real-valued function defined on $[0,1]$ such that ϕ, ϕ' and ϕ'' are continuous on $[0,1]$. Prove that

$$\int_0^1 \cos x \ \frac{x\phi'(x) - \phi(x) + \phi(0)}{x^2} dx < \frac{3}{2} \left\| \phi'' \right\|_\infty ,$$

where $\|\phi''\|_\infty := \sup_{x\in[0,1]} |\phi''(x)|$. (Note, the constant $\frac{3}{2}$ in the inequality may not be the smallest possible constant).

(6) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued continuous functions defined on $[0,1]$ such that $\int_0^1 |f_n(y)|dy \leq 3$, for all $n \in \mathbb{N}$. Define $g_n : [0,1] \to \mathbb{R}$ by:

$$g_n(x) = \int_0^1 \sqrt{x+y}\, f_n(y)dy.$$

Prove that $\{g_n\}_{n=1}^{\infty}$ contains a subsequence that converges uniformly on $[0,1]$.