Analysis Qualifier Examination

Tuesday, June 1, 2010, 12:30 – 6:30pm, Avery 119

• Work 5 out of 6 problems. • Each question is worth 20 points. • Write on one side of the paper only.

Question 1.

a) Suppose f, g, and h are bounded real valued functions on $[0,1]$ with $f \leq g \leq h$. If f and h are Riemann integrable with $\int_0^1 f(x)\,dx = \int_0^1 h(x)\,dx$, prove that g is Riemann integrable.

b) Suppose $f \leq g$ on $[0,1]$ and f and g are known to be continuous. Show that if $\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx$ then $f = g$.

Question 2.

a) Prove that if (X,d) is a compact metric space and $f : X \to \mathbb{R}$ is a continuous function, then there exists $x_0 \in X$ such that $f(y) \leq f(x_0)$ for every $y \in X$.

b) Using the open covers definition, prove that

$$S := \{1/n \,:\, n \in \mathbb{Z}\} \cup \{0\}$$

is compact.

Question 3.

a) Let $(a_k)_{k\geq 1}$ be a sequence of real numbers such that the series $\sum_{k=1}^{\infty} a_k$ converges, and the series $\sum_{k=1}^{\infty} a_k a_{k+1}$ diverges. Prove that $\sum_{k=1}^{\infty} a_k$ does not converge absolutely.

b) Give an example of a sequence $(a_k)_{k\geq 1}$ of real numbers with this property. (I.e., such that the series $\sum_{k=1}^{\infty} a_k$ converges, and the series $\sum_{k=1}^{\infty} a_k a_{k+1}$ diverges.)

Question 4. Suppose that $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are sequences of real numbers that satisfy, for all $n > 1$,

$$a_n = \frac{1}{2}(a_{n-1} + b_{n-1}).$$

a) Prove that if $a_n \to 0$ then $b_n \to 0$.

b) Prove that if $b_n \to 0$ then $a_n \to 0$.

Question 5. Suppose $f, g : (0,1) \to \mathbb{R}$ are differentiable and satisfy the following:

$$\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x),$$

$$\lim_{x\to 0} f'(x) = A, \quad \lim_{x\to 0} g'(x) = B \text{ and } B \neq 0.$$

a) Prove that there exists $\delta > 0$ such that $|g(x)| > 0$ for every $x \in (0,\delta)$.

b) Prove the following version of l'Hospital's Rule:

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

Hint: Start with the special case

$$\lim_{x\to 0} \frac{f(x)}{x} = A.$$

Question 6. Suppose $f : [0,1] \to \mathbb{R}$ is continuous and $f(0) \neq f(1)$. Let $f_n(x) := f(x^n)$.

a) Prove that f_n does not converge uniformly on $[0,1]$.

b) Prove that $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = f(0)$.