Instructions: Work 5 out of the 6 questions. Each question is worth 20 points. Write on one side of the paper only. You can (and *should*) quote appropriate theorems as long as doing so does not trivialize the problem.

1. (a) Prove, from the definition, that the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is continuous.

   (b) Prove, using the open covering definition of compactness, that the set $\{0, 1\} \cup \left\{\frac{1}{n}, \frac{n-1}{n} : n = 1, 2, \ldots\right\}$ is compact.

2. (a) Define what it means for $f : [0, 1] \to \mathbb{R}$ to be Riemann integrable on $[0, 1]$.

   (b) Prove that if $f : [0, 1] \to \mathbb{R}$ is continuous then the function $F : [0, 1] \to \mathbb{R}$ defined by
   $$F(t) = \int_0^t f(x)\,dx,$$
   is differentiable, and has $F'(t) = f(t)$ for all $t \in (0, 1)$.

   (c) Show that the result of the previous part does not hold for all Riemann integrable $f : [0, 1] \to \mathbb{R}$.

3. The parts of this question are unconnected.

   (a) Suppose that $f : [-1, 1] \to \mathbb{R}$ is continuous and satisfies $f(-1) < 0 < f(1)$. Prove directly that there exists $c \in (-1, 1)$ such that $f(c) = 0$. No form of the Intermediate Value Theorem may be used without proof.

   (b) Let $(a_n)_{n\geq 1}$ be a sequence of non-increasing, non-negative, continuous functions, $a_n : \mathbb{R} \to \mathbb{R}$. Prove that if $\sum_{n=1}^{\infty} a_n(x)$ converges for all x, then the function
   $$g(x) = \sum_{n=1}^{\infty} a_n(x)$$
   is continuous.

4. The parts of this question are unconnected.

   (a) Determine the values of $x \in \mathbb{R}$ for which $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{1 + n\,|x|^n}$ converges, justifying your answer carefully.

   (b) Let $f : \mathbb{R} \to (0, \infty)$ be non-decreasing and suppose that $\liminf_{n\to\infty}(f(n+1) - f(n)) > 0$. Prove that
   $$\limsup_{x\to\infty} \frac{f(x)}{x} > 0.$$

5. (a) An infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfies the differential equation

$$f^{(3)}(x) = f(x). \qquad (*)$$

Prove that there exists $M = M(R)$ such that for all x with $|x| \leq R$ and all $j \geq 0$ we have

$$\left|f^{(j)}(x)\right| \leq M.$$

(b) Suppose now that in addition to f satisfying (*) we have

$$f(0) = 1, f'(0) = f''(0) = 0.$$

Using Taylor's theorem, or otherwise, prove that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}.$$

6. Recall that Dini's Theorem states the following. If $K \subset \mathbb{R}$ is compact and $f : K \to \mathbb{R}$, $f_n : K \to \mathbb{R}$ are continuous, with $f_{n+1}(x) \leq f_n(x)$ for all $n \in \mathbb{N}$, and in addition $f_n(x) \to f(x)$ pointwise on K then $f_n \to f$ uniformly on K.

(a) Prove Dini's Theorem.

(b) Give an example to show that the compactness of K is a necessary condition for Dini's Theorem. (You don't need to have a proof of (a) to do (b).)