Real Analysis Qualifying Examination–Math 825/826

May 31, 2012

• **Work 5 out of 6 problems.** • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order. • Write your exam ID Code on every paper. • $\mathbb{R}$ denotes the real numbers, $\mathbb{Q}$ denotes the rational numbers, and $\mathbb{N}$ the natural numbers. • You can (and should) quote appropriate theorems as long as doing so does not trivialize the problem.

(1) (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 0$. Prove that f is differentiable at $x = 0$ if and only if there is a function $g : \mathbb{R} \to \mathbb{R}$ that is continuous at $x = 0$ and satisfies $f(x) = xg(x)$ for all $x \in \mathbb{R}$.

(b) Suppose $f : [0,1] \to [0,1]$ is continuous. Show that there exists $0 \leq x \leq 1$ so that $x + f(x) = 1$.

(2) Let (X, ρ) be a metric space. Define $\sigma : X \times X \to \mathbb{R}$ by

$$\sigma(x,y) := \min\{1, \rho(x,y)\}, \quad \text{for each } x, y \in X.$$

(a) Show that σ is a metric on X.

(b) Let $(x_n)_{n=1}^{\infty}$ be a sequence in X. Show that $(x_n)_{n=1}^{\infty}$ is convergent in (X, ρ) if and only if it is convergent in (X, σ).

(3) (a) Let A be a connected set in $\mathbb{R}^n$ and let $cl(A)$ denote the closure of A. Prove by definition only that any subset B with $A \subset B \subset cl(A)$ must be connected.

(b) Let $k \to q_k$ be a bijection from the strictly positive natural numbers to the rational numbers. Show that

$$\mathbb{R} \setminus \bigcup_{k=1}^{\infty} \left(q_k - \frac{1}{k^2}, q_k + \frac{1}{k^2} \right) \neq \emptyset.$$

(4) (a) Let f and g be continuous functions on $[a,b]$ with $g(x) \geq 0$ for all $x \in [a,b]$. Prove that there exists $x \in [a,b]$ such that $\displaystyle\int_a^b f(t)g(t)dt = f(x) \int_a^b g(t)dt$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x$ and

$$g(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is not Riemann-Stieltjes integrable with respect to g on the interval $[0,1]$.

(5) Let $f_n : [0,1) \to \mathbb{R}$ be the function defined by

$$f_n(x) := \sum_{k=1}^{n} \frac{x^k}{1+x^k}.$$

(a) Prove that f_n converges to a function $f : [0,1) \to \mathbb{R}$.
(b) Prove that for every $0 < a < 1$ the convergence is uniform on $[0,a]$.
(c) Prove that f is differentiable on $(0,1)$.

(6) (a) Determine the interval of convergence for $\displaystyle\sum_{n=0}^{\infty} 3^{(-1)^n - n} x^{2n+1}$.

(b) Let $f : [0,1] \to \mathbb{R}$ be continuous and suppose $\displaystyle\int_0^1 f(x)x^{2n+1}dx = 0$ for all integers $n \geq 0$. Prove that $f(x) = 0$ for all $x \in [0,1]$.

The End