**Masters and Ph.D. Qualifying Exam** **Analysis: Math 825/826**

Thursday, June 5, 2014, 12:00 - 6:00p.m.

• Work 5 out of 6 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.
• Do not interpret a problem in such a way that it becomes trivial.

(1) Define $\alpha : [-1,1] \to \mathbb{R}$ by

$$\alpha(x) := \begin{cases} -1, & x \in [-1,0]; \\ 1, & x \in (0,1]. \end{cases}$$

Let $f : [-1,1] \to \mathbb{R}$ be a function that is uniformly bounded on $[-1,1]$ and continuous at $x = 0$, but not necessarily continuous for $x \neq 0$. Prove that $f$ is Riemann-Stieltjes integrable with respect to $\alpha$ over $[-1,1]$ and that

$$\int_{-1}^{1} f(x)\, d\alpha(x) = 2f(0).$$

(2) (a) State the Weierstrass approximation theorem.

(b) Let $[a,b] \subset \mathbb{R}$ be given. Suppose that $f : [a,b] \to \mathbb{R}$ is continuous and that

$$\int_a^b f(x)x^n\, dx = 0 \quad \text{for each } n = 0,1,2,3,\ldots.$$

Prove that $f(x) = 0$ for all $x \in [a,b]$.

(3) Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ with the following property: there are numbers $0 < \lambda \leq \Lambda$ such that for all integers $n > 0$

$$\lambda x^n \leq f_n(x) \leq \Lambda x^n \text{ for each } x \in [0,1].$$

(a) Prove that the series $\displaystyle\sum_{n=1}^{\infty} f_n(x)(1-x)$ converges pointwise **but not** uniformly on $[0,1]$.

(b) Prove that the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n f_n(x)(1-x)$ converges uniformly on $[0,1]$.

(4) Fix $x_0 \in \mathbb{R}$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable; i.e. has continuous derivatives on $\mathbb{R}$ of all orders. Let $k \in \{0,1,2,\ldots\}$ be given.

(a) Provide the formula for $P_k : \mathbb{R} \to \mathbb{R}$, the $k$-th order Taylor polynomial for $f$ centered at $x_0$.

(b) Suppose that $Q : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree $k$ satisfying

$$\lim_{x \to x_0} \frac{f(x) - Q(x)}{(x - x_0)^k} = 0.$$

Prove that $Q = P_k$.

(5) Let $(X, \rho)$ be a metric space.

(a) Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences from $X$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} y_n = y_0$ for some $x_0, y_0 \in X$. Argue that

$$\lim_{n\to\infty} \rho(x_n, y_n) = \rho(x_0, y_0).$$

(For this part **do not** use, without proof, the fact that $\rho$ is continuous.)

(b) Under the assumption that $(X, \rho)$ is compact, verify that there exist $a, b \in X$ such that

$$\rho(a,b) = \sup\{\rho(x,y) \,:\, x, y \in X\}.$$

(For this part you may assume, without proof, that $\rho$ is continuous.)

(6) (a) Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence from $\mathbb{R}$. State the definitions of $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

(b) Let $\{a_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers in $(0,1)$. Compute, with justification, the numerical values of $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.