Math 825-826 Qualifying Exam Spring 2013

Work any six of the given problems, clearly marking the one problem you do not want graded.

1. (20 points)

   (a) Define $\lim_{n\to\infty} s_n = L$ and use your definition to prove $\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n}) = \frac{1}{2}$.

   (b) Give the ϵ definition of $\lim_{x\to-\infty} f(x) = L$ and use your definition to prove that $\lim_{x\to-\infty} \frac{x-1}{x} = 1$.

2. (20 points) Assume $f : [a,b] \to \mathbb{R}$ is bounded. Define f is Riemann integrable on $[a,b]$ in terms of upper and lower Riemann sums. Prove that f is Riemann integrable on $[a,b]$ if and only if given any $\epsilon > 0$, there is a partition P_ϵ of $[a,b]$ such that $U(f,P_\epsilon) - L(f,P_\epsilon) < \epsilon$.

3. (20 points) Assume $f : \mathbb{R} \to \mathbb{R}$ is continuous. Let $\{\delta_k\}_{k=1}^\infty$ be a decreasing sequence of real numbers with limit 0. Define the sequence $\{h_k\}_{k=1}^\infty$ by

$$h_k(x) = \begin{cases} f(k), & x > k \\ f(x), & |x| \le k \\ f(-k), & x < -k. \end{cases}$$

   Then define the sequence of the so called "integral means" $\{g_k\}_{k=1}^\infty$ by for each $x \in \mathbb{R}$ $g_k(x) = \frac{1}{2\delta_k}\int_{x-\delta_k}^{x+\delta_k} h_k(t)\ dt$. Prove each of the following: each g_k is continuous on $\mathbb{R}$ and $|g_k(x)| \le M_k := \max\{|f(x)| : |x| \le k\}$, on $\mathbb{R}$, each g_k is continuously differentiable and $|g_k'(x)| \le \frac{M_k}{\delta_k}$ on $\mathbb{R}$. Prove that $\lim_{k\to\infty} g_k(x) = f(x)$ uniformly on compact subsets of $\mathbb{R}$. Finally, if

$$f(x) = \begin{cases} 1, & x \ge 0 \\ -1, & x < 0 \end{cases}$$

   find $g_k(x)$.

4. (20 points) Show that the infinite series $\sum_{n=0}^\infty x^n 2^{-nx}$ converges uniformly on $[0,B]$ for any $B > 0$. Does this series converge uniformly on $[0,\infty)$? Prove your answer.

5. (20 points)

   (a) Let (M,d) be a metric space, then if $x_0 \in M$ we define the open ball about x_0 of radius $r > 0$ by $B_r(x_0) = \{x \in M : d(x,x_0) < r\}$. Prove that $B_r(x_0)$ is an open set in M. (Use the basic ϵ definition of an open set in Donsig's book.)

   (b) Let (X,ρ) be a complete metric space, assume[1] $A \subset X$ and $F : X \to X$. Assume there is a $k > 0$ such that $\rho(F(a),F(b)) \le k \cdot \rho(a,b)$ for all $a,b \in X$. Suppose further $A \subset F(A)$. Provide, with proof, a discription of A if $k < 1$. Can anything be said about the set A if $k \ge 1$? (prove or disprove your answer.)

6. (20 points) Prove Dini's Theorem: Assume $f_n : [a,b] \to \mathbb{R}$, is a decreasing sequence of continuous functions with $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in [a,b]$, where the limit function f is continuous on $[a,b]$. Show that $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly of $[a,b]$. Give an example to show that we can not replace $[a,b]$ in Dini's Theorem by (a,b). Give an example that shows that we can not delete the assumption in Dini's Theorem that the limit function is continuous.

7. (20 points) Prove that if $f \in R(\alpha)$ on $[a,b]$ and $\alpha \in C^1[a,b]$, then the Riemann integral $\int_a^b f(x)\alpha'(x)dx$ exists and

$$\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx.$$

[1] The statement should have included an additional assumption that A is **bounded**. Without it the question is too broad.