Masters & Ph.D. Qualifying Exam **Analysis: Math 825/826**

January 18, 2017 12:00-6:00p.m., Avery Hall 351

• Work 5 questions out 6. • Each problem is worth 20 points.
• Parts of each problem don't necessarily carry the same weight, and they can be unrelated. • Do not interpret a problem in such a way that it becomes trivial. • Write on one side of the paper only and hand your work in order.

(1) Suppose that $f:\mathbb{R}\to\mathbb{R}$ is given by: $f(x)=\dfrac{x^3}{1+x^2}$. By using the ϵ–δ definition only, prove that f is uniformly continuous on $\mathbb{R}$. Hint: write $f(x)=x-\dfrac{x}{1+x^2}$.

(2) Let (X,d) be a metric space. A function $f:X\to\mathbb{R}$ is said to be lower semicontinuous at $x_0\in X$ if, for every $\epsilon>0$ there exists $\delta>0$ such that for all $x\in X$ with $d(x,x_0)<\delta$, then $f(x)-f(x_0)>-\epsilon$. Prove:
f is lower semicontinuous at $x_0\in X$ **if and only if** $\liminf\limits_{n\to\infty} f(x_n)\geq f(x_0)$ whenever $\{x_n\}_{n=1}^{\infty}\subset X$ satisfies $\lim\limits_{n\to\infty} x_n=x_0$ in X.

(3) Let (X,d) be a **compact** metric space. Suppose that $f_n:X\to[0,\infty)$ is a sequence of continuous functions with $f_n(x)\geq f_{n+1}(x)$, for all $n\in\mathbb{N}$ and all $x\in X$, and such that $f_n\longrightarrow 0$ point-wise on X. Prove $\{f_n\}_{n=1}^{\infty}$ converges to zero uniformly on X.

(4) Let $f:[-1,1]\longrightarrow\mathbb{R}$ be a function such that $f\in C^2([-1,1])$, i.e., f is twice-continuously differentiable on $[-1,1]$, and f''' exists on $(-1,1)$ with $|f'''(x)|\leq 1$ for all $x\in(-1,1)$. Let:

$$a_n=n\Big(f(\frac{1}{n})-f(\frac{-1}{n})\Big)-2f'(0).$$

Prove that the series $\sum_{n=1}^{\infty}a_n$ converges absolutely.

(5) Suppose that $f:[0,1]\to\mathbb{R}$ is a continuous function which satisfies, for $n=0,1,2,...$,

$$\int_0^1 f\left(\sqrt[2n+1]{x}\right)dx=0.$$

Prove that $f(x)=0$ for all $x\in[0,1]$.

(6) For each $n\in\mathbb{N}$, let $f_n:[0,\infty)\longrightarrow\mathbb{R}$ be given by: $f_n(x)=\dfrac{\sin(nx)}{1+nx}$.
(a) Prove f_n converges point-wise on $[0,\infty)$ and find the point-wise limit f.
(b) Show that $f_n\longrightarrow f$ uniformly on $[a,\infty)$ for every $a>0$, but the convergence is not uniform on $[0,\infty)$.