January-2018 Masters & Ph.D. Qualifying Exam **Analysis: Math 825/826**

• Work 5 questions out 6. • Each problem is worth 20 points.
• Parts of each problem don't necessarily carry the same weight, and they can be unrelated. • Do not interpret a problem in such a way that it becomes trivial. • Write on one side of the paper only. • Hand in your work in order.

(1) Let α be a monotonically increasing function on an interval $[a, b]$ and assume α' is Riemann integrable in $[a, b]$. Let f be a bounded real function on $[a, b]$. Prove that $f \in \mathcal{R}(\alpha)$ implies $f\alpha' \in \mathcal{R}[a, b]$, and

$$\int_a^b f d\alpha = \int_a^b f\alpha' dx.$$

(2) Let X be a metric space in which every infinite subset has a limit point. Prove that there is a countable subset D of X that is dense in X.

(3) Let $\{a_n'\}$ be any rearrangement of an infinite sequence $\{a_n\}$. Assume $\sum_{n=1}^{\infty} a_n$ converges absolutely. Prove $\sum_{n=1}^{\infty} a_n' = \sum_{n=1}^{\infty} a_n$.

(4) Suppose $f_n \to f$ uniformly on $[0, \infty)$ and $\lim_{x\to\infty} f_n(x)$ exists for all n. Prove that $\lim_{x\to\infty} f(x)$ exists and

$$\lim_{x\to\infty} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to\infty} f_n(x).$$

(5) Let $f : [a, b] \to \mathbb{R}$. Suppose $f \in BV[a, b]$. Prove f is the difference of two increasing functions.

(6) Suppose that $f : [0, 2] \to \mathbb{R}$ is continuous on $[0, 2]$, differentiable on $(0, 2)$, satisfying $f(0) = f(2) = 0$ and $f(c) = 1$ for some $c \in (0, 2)$. Prove that there is an $x \in (0, 2)$ such that $|f'(x)| > 1$.

End