June-2017 Masters & Ph.D. Qualifying Exam **Analysis: Math 825/826**

• Work 5 questions out 6. • Each problem is worth 20 points.
• Parts of each problem don't necessarily carry the same weight, and they can be unrelated. • Do not interpret a problem in such a way that it becomes trivial. • Write on one side of the paper only. • Hand in your work in order.

(1) Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+nx}$, $x \in \mathbb{R}$. Justify your answers to the following questions:
(a) Does the series converge at $x = 0$?
(b) Does the series converge on $(0, \infty)$?
(c) Does the series converge uniformly on $(0, \infty)$?
(d) Does the series converge uniformly on $[a, \infty)$ for any $a > 0$.

(2) Prove: $f \in \mathcal{R}(\alpha)$ on $[a, b]$ $\iff$ For any $a < c < b$, $f \in \mathcal{R}(\alpha)$ on $[a, c]$ and on $[c, b]$. In addition, if either condition holds, then we always have

$$\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha$$

(3) Let $f(x) = \sum_{n=0}^{\infty} a_n(1-x)x^n$, where $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$. Assume $\lim_{n\to\infty} a_n = a \in \mathbb{R}$.
(a) Prove: $f(x)$ converges absolutely for $|x| < 1$.
(b) Prove: $\lim_{x\to 1^-} f(x) = a$.

(4) Let $f \in C[0, 1]$. Prove that there is a sequence of polynomials, $\{p_n\}$, of even degree monomials (i.e. every term in p_n is of the form $a_k x^{2k}$ for integer $k \geq 0$) that converges to f uniformly on $[0, 1]$.

(5) Assume $\{f_n\}$ is a sequence of monotonically increasing functions on $[a, b]$, and $f_n \to f$ point-wise. Prove that if f is continuous then the convergence $f_n \to f$ is uniform on $[a, b]$.

(6) Let (X, d) be a metric space. A function $f : X \to \mathbb{R}$ is said to be lower semi-continuous (l.s.c.) if $f^{-1}(a, \infty) = \{x \in X : f(x) > a\}$ is open in X for every $a \in \mathbb{R}$. Analogously, f is upper semi-continuous (u.s.c.) if $f^{-1}(-\infty, b) = \{x \in X : f(x) < b\}$ is open in X for every $b \in \mathbb{R}$.
(a) Prove that a function $f : X \to \mathbb{R}$ is continuous $\iff$ f is both l.s.c and u.s.c.
(b) Prove that f is lower semi-continuous (l.s.c.) $\iff$ $\liminf_{n\to\infty} f(x_n) \geq f(x)$ whenever $\{x_n\}_{n=1}^{\infty} \subset X$ such that $x_n \to x$ in X.

End