Masters and Ph.D. Qualifying Exam **Analysis: Math 825/826**

Wednesday, June 1, 2022.

• Work 5 out of 6 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.
• Do not interpret a problem in such a way that it becomes trivial.

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(1) (a) Let $f_n(x) = nx(1-x)^n$ for $x \in [0,1]$. Prove that $\{f_n\}$ converges pointwise and determine if it converges uniformly on $[0,1]$. Is $\{f_n\}$ equicontinuous? Clearly motivate your answer.
(b) Prove that in general, if $\{f_n\}_{n\geq 1}$ is an equicontinuous sequence of functions on a compact interval and $f_n \to f$ pointwise, then the convergence is uniform.

(2) Let (X,ρ) be a metric space. Suppose that $x_0 \in X$. For each $\varepsilon > 0$, set

$$E_\varepsilon := \{x \in X : \rho(x, x_0) \geq \varepsilon\}.$$

Suppose that $f : X \to \mathbb{R}$ is continuous and $f(E_\varepsilon)$ is compact for all $\varepsilon > 0$. Prove that $f(X)$ is compact.

(3) Consider the sequence $\{x_n\}_{n\geq 1}$ defined by $0 < x_1 < 1$ and $x_{n+1} = 1 - \sqrt{1-x_n}$ for $n = 1, 2, ...$ Show that $x_n \to 0$ as $n \to \infty$. Also, show that $\dfrac{x_{n+1}}{x_n} \to \dfrac{1}{2}$.

(4) Use the Riemann condition to show that $f \in \mathcal{R}_\alpha[0,3]$ where $f(x) = \ln(2x+1)$ and

$$\alpha(x) = \begin{cases} x+2, & 0 \leq x \leq 2 \\ 3x-1, & 2 < x \leq 3. \end{cases}$$

Compute the value of the Riemann-Stieltjes integral $\displaystyle\int_0^3 f(x)d\alpha$.

(5) Find the domain of convergence and the sum of the series

$$\sum_{n\geq 0} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Show how one may use the sum of the series to provide an approximation for π up to three decimals. Be sure to provide all technical details.

(6) Show that

$$d_1(f,g) := \int_0^1 |f(x) - g(x)|\, dx \quad \text{and} \quad d_\infty(f,g) := \text{lub}_{x\in[0,1]}|f(x) - g(x)|$$

are metrics on $C[0,1]$, but are not equivalent on $C[0,1]$.