Analysis Qualifying Examination

Wednesday, January 12, 2022, Noon–5:00pm

INSTRUCTIONS: Work 5 of the following 6 problems. Write on only one side of each page. Each problem is worth 20 points.

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1) Suppose $a, b \in \mathbb{R}$ with $a < b$ and let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function such that $f'\colon [a,b] \to \mathbb{R}$ is continuous. Show that for every $\epsilon > 0$, there is a $\delta > 0$ such that for every $x, y \in [a,b]$ with $|x - y| < \delta$, we have

$$\Big|\frac{f(x) - f(y)}{x - y} - f'(x)\Big| < \epsilon.$$

2) Let X and Y be metric spaces and let $f\colon X \to Y$ be a continuous bijection.

a) Show by example that $f^{-1}\colon Y \to X$ need not be continuous.

b) Show that if X is compact, then $f^{-1}\colon Y \to X$ is continuous.

3) Compute, with proof, $\lim\limits_{k\to\infty} \sum\limits_{n=1}^{\infty} n^{-k}$.

4) a) Let $f\colon [1,2] \to \mathbb{R}$ be a continuous function. If $\int_1^2 x^{-n} f(x)\, dx = 0$ for all integers $n \geq 0$, show that $f = 0$.

b) Let $g\colon [1,2] \to \mathbb{R}$ be a differentiable function such that $g'\colon [1,2] \to \mathbb{R}$ is continuous. If $\int_1^2 x^{-n}\, dg(x) = 0$ for all integers $n \geq 0$, show that g is constant.

5) Prove the following special case of Dini's Theorem: if $\big(f_n\colon [0,1] \to \mathbb{R}\big)_{n=1}^{\infty}$ is decreasing sequence of continuous functions such that $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$, then $(f_n)_{n=1}^{\infty}$ converges uniformly to 0. (You should not use any form of Dini's theorem without proof.)

6) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function.

a) Show $\lim\limits_{n\to\infty} \int_0^1 f(x^{1/n})\, dx = f(1)$.

b) If $f(x) > 0$ for all $x \in [0,1]$, show $\lim\limits_{n\to\infty} \int_0^1 f(x)^{1/n}\, dx = 1$.