Analysis Qualifying Examination

Tuesday, May 25, 2021, Noon–5:00pm

INSTRUCTIONS: Work 5 of the following 6 problems. Write on only one side of each page. Each problem is worth 20 points.

1) Suppose that X is a compact metric space, $(x_n)_{n=1}^\infty \subseteq X$, $x \in X$, and every convergent subsequence of $(x_n)_{n=1}^\infty$ converges to x. Show that $(x_n)_{n=1}^\infty$ converges to x.

2) a) Show that if $f\colon [a,b] \to \mathbb{R}$ is a differentiable function and $f'\colon [a,b] \to \mathbb{R}$ is bounded, then f has bounded variation.

b) Construct, with proof, a continuous function $g\colon [a,b] \to \mathbb{R}$ which does not have bounded variation.

3) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function such that there is no $x \in \mathbb{R}$ with $f(x) = f'(x) = 0$. Show that if $a, b \in \mathbb{R}$ with $a < b$, then $\{x \in [a,b] : f(x) = 0\}$ is a finite set.

4) Let X be a metric space. A function $f\colon X \to \mathbb{R}$ is called *lower semicontinuous* if for every convergent sequence $(x_n)_{n=1}^\infty \subseteq X$, we have

$$f\big(\lim_{n\to\infty} x_n\big) \le \liminf_{n\to\infty} f(x_n).$$

a) Let $A \subseteq X$ be a set and define

$$\chi_A\colon X \to \mathbb{R}\colon x \mapsto \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

Show that χ_A is lower semicontinuous if and only if A is open.

b) Show that if X is a compact metric space and $f\colon X \to \mathbb{R}$ is lower semicontinuous, then there is a $c \in X$ such that

$$f(c) = \inf\{f(x) : x \in X\}.$$

5) Suppose that $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are sequences of strictly positive real numbers such that $\displaystyle\sum_{n=1}^\infty b_n$ converges, and suppose that for each integer $n \ge 1$, we have $\dfrac{a_{n+1}}{a_n} \le \dfrac{b_{n+1}}{b_n}$. Show that $\displaystyle\sum_{n=1}^\infty a_n$ converges.

6) A function $f\colon [0,1] \to \mathbb{R}$ is called *Lipschitz* if there is a constant $C > 0$ such that for all $x, y \in [0,1]$, we have

$$|f(x) - f(y)| \le C|x - y|.$$

Let $\mathrm{Lip}([0,1])$ denote the set of all Lipschitz functions $[0,1] \to \mathbb{R}$ with the uniform metric

$$d_\infty(f,g) = \sup\big\{|f(t) - g(t)| : 0 \le t \le 1\big\}, \qquad f, g \in \mathrm{Lip}([0,1]).$$

Show that $\mathrm{Lip}([0,1])$ is a countable union of compact sets.