# Analysis Qualifying Examination

Wednesday, June 3, 2020, 11:45am–5:00pm

**INSTRUCTIONS:** Work 5 of the following 6 problems. Write on only one side of each page. Each problem is worth 20 points.

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**1)** Suppose for each $n \in \mathbb{N}$, $f_n : [0,1] \to \mathbb{R}$ is continuous and satisfies $\sup\{|f_n(x)| : x \in [0,1]\} \leq 2020$. Put $g_n(x) = \displaystyle\int_0^x f_n(t)\,dt$. Prove that there is a subsequence of $(g_n)$ which converges uniformly on $[0,1]$.

**2)** Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and that $f''(t) < 0$ for every $t \in \mathbb{R}$. Prove that for any real numbers satisfying $a < x < b$,

$$(f(x) - f(a))(b - a) > (f(b) - f(a))(x - a).$$

**3)** Suppose $(X, d)$ is a compact metric space.

a) Let $E$ be a closed subset of $X$ and for $x \in X$, define $\operatorname{dist}(x, E) = \inf\{d(x, e) : e \in E\}$. Show that if $x \in X \setminus E$, then $\operatorname{dist}(x, E) > 0$.

b) Suppose $f : X \to X$ is an isometry, that is, for every $x, y \in X$, $d(x, y) = d(f(x), f(y))$. Prove that $f(X) = X$.

**4)** Suppose $f : [0, \infty) \to \mathbb{R}$ is a uniformly continuous function. Prove there are real numbers $A$ and $B$ such that for every $x \in [0, \infty)$,

$$|f(x)| \leq A + Bx.$$

**5)** The following questions are not related.

a) Let $\alpha(x) = \begin{cases} x & x < 0 \\ e^x & x \geq 0. \end{cases}$ Evaluate $\displaystyle\int_{-1}^{1} x\,d\alpha$.

b) Suppose $f : [0,1] \to \mathbb{R}$ is continuous. Evaluate $\displaystyle\lim_{n\to\infty} n \int_0^1 x^n f(x)\,dx$. Give a complete proof of your answer.

**6)** Suppose $f : [0,1] \to [0,1]$ is a function such that $f(0) = 0$, $f(1) = 1$, has a continuous derivative, and $f'(x) > 0$ for every $x \in [0,1]$.

Prove there exists a sequence $P_n$ of polynomials such that $P_n$ converges uniformly to $f$ and such that for every $n$, $P_n$ is strictly increasing, $P_n(0) = 0$, and $P_n(1) = 1$.